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# A novel case of practical exponential observer using extended Kalman filter

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**ABSTRACT.** This technical note presents a case of practical exponential observer using extended Kalman filter independent of certain restrictions, such as online check and estimation error of initial state. Recursive state estimation is usually a challenge for discrete-time nonlinear system in terms of computation cost. Extended Kalman filter (EKF) is attractive with its simplicity since it is considered as an exponential observer given the above restrictions. However, those restrictions are so mathematically complicated that EKF cannot be practical in estimation. A novel case for an exponential observer using EKF is proposed which is independent of such restrictions. However, these restrictions are proved to be unnecessary in the case. The proposed case is illustrated by a navigation system scenario. The validity of the case is demonstrated by a numerical simulation experiment. The system is deterministic.

**INDEX TERMS.** discrete-time nonlinear system, extended Kalman filter, exponential observer, restrictions, spectral norm

## I. INTRODUCTION

The recursive state estimation is a typical problem of discrete-time dynamic systems, especially in state estimation for navigation system. The estimation problem from noise or incomplete measured data became an issue of worldwide research interest in the last several decades. However, it is not easy to find a practical estimation method because of requirements of low computation costs and none parameter regulation. The basic solution on the basis of the Bayesian approach is provided by the Bayesian recursive relations (BRRs). The solution is to compute the probability density functions (pdfs) of the state. And these pdfs are conditioned by the measurements but fully describe the immeasurable state. The closed form solution to the BRRs is valid for some of special cases [1]. One typical example is for a linear Gaussian system, which introduces the famous Kalman filter (KF). However, it is different in other cases for nonlinear systems. Some approximation methods are usually applied to approximate the nonlinear model. One popular kind of approximation is named as global method. It is derived from a kind of approximation of the BRRs with the generation of conditional pdf of the state. For example, these are the particle filter (PF)[2,3], the point-mass method [4], the Gaussian sum method, or the ensemble Kalman filter [5].

The disadvantage of these methods is big computational cost and thus it is necessary to find a different method with relatively low computation consumption

Relatively, the other approximation is named as local method. It is based on an approximation of a system model. One obvious benefit of this method is that it is possible for the KF design technique to be used for the BRRs solution; that is, not only the covariance matrix but also conditional mean is computed instead of the conditional pdf. In fact, this approximation of the model is rough. Together with a posteriori estimates, it is induced that the validity of the estimates is local. Consequently, it is hard to guarantee convergence of the estimates of the local methods. Nevertheless, there is one advantage using the local methods that is the simplicity of the BRRs solution. For example, the extended Kalman filter (EKF) is a local method and is based on function approximation using the Taylor or Stirling expansions. These presented Taylor expansions are a kind of important tools which will be used in the proving process related to our presented work (as shown in Section III) [6]–[9]. They have pointed out EKF is a second order filter and is usually used to solve nonlinear problems. The problem of EKF is that it is not reliable with poor initiation. Divided difference filters are different from general EKF. They are used for approximation of the pdfs and the state estimates are

represented by a set of selected weighted points [10]~[13]. This approach uses the unscented transform (UT). The unscented Kalman filter, the Gauss-Hermite filter, or the cubature Kalman filter all belong to this approach. This kind of approach is also a local method which gives a modified EKF frame but remains to be vulnerable to weak initial conditions. The fact that modified EKF cannot be satisfactory may give us a hint of modifying the system itself (as illustrated in Section II). The Moving Horizon Estimator(MHE) is an alternative to recursive estimators as it takes not only the nonlinearity of the process model but also different kinds of constraints into account [14]. In contrast, the calculation of MHE may result in a computational load because of repeat optimization. Thus it is proved repeat optimization is not available method in the practical application. Receding-horizon Nonlinear Kalman (RNK) filter has been proposed for the recursive state estimation to address the computational constraints which exists in MHE [15]. The main contribution is RNK formulation gets computational gain because of its standard EKF framework. However, the convergence of RNK has not been analyzed and it seems to be further exploring. Recently, extended state observer (ESO) become an attracted filter due to regarding the disturbances of the system as a new state. For considering the fast varying disturbances, a modified ESO has been proposed with well performance in [16]. Considering the estimation for sensor networks, a distributed interval type-2(IT2) fuzzy filter model has been proposed with upper and lower bounds in membership functions [17]. It is useful in the estimation for static systems. In addition, the filter design for fault detection(FD) has also been investigated for nonhomogeneous Markovian jump systems [18]. The full FD filters have been designed desirably by a Takagi-Sugeno Fuzzy approach.

Regarding the convergence of the EKF for discrete-time nonlinear systems, [19] presents a significant role forward in exponential stability analysis. The contribution in this paper is that it is EKF for discrete-time nonlinear system is proved to be regarded as an exponential observer given certain conditions; that is, the dynamics of the estimation error are exponentially stable. This result points out EKF can be an exponential observer and it makes EKF attractive for estimation of nonlinear system. Thus we go back to consider the EKF as a potential algorithm basic frame. In similar to the results before, it is also pointed out that the conditions have to be checked online as well as the estimation error of initial state small enough. Actually, it is not practical since usually it is hard for us to select an initial state with small estimation error as well as check online. Thus practical exponential observer is actually being pursuing for the sake of engineering realization. Then one important question has to be asked, is there any case that EKF might become an exponential observer independent of those restrictions? This question has not been answered not clearly and thus motivates the main objective of this note: To design a practical exponential observer considering low computation costs and none

regulation of initial condition.

To address this problem, we can deduce a further solution through the proving process of exponential observer in [19]. As result, a novel case for state of estimation of navigation system is proposed in this technical note. The proposition is proved on the basis results in [19]. We prove there is indeed such an exponential observer using EKF. The proposed case of the observer consists of two parts. First, the linearized system matrix in nonlinear system (1) is modified into the frame as shown in Theorem 3.1 which is proved in Section III. Then a traditional EKF is used to obtain the estimates of the nonlinear system. Considering a position estimate scenario, one typical nonlinear discrete-time system is introduced through a navigation system. Only a single range is available for measurement in the system. Generally, it is clear the EKF of this nonlinear system is not exponential stable given none online check and poor estimation of initial state. In contrast, we show the estimates can convert quickly in the proposed case and do not need any such restrictions, neither online check nor small estimation error of initial state. We also prove the key is to set the norm of the 'system matrix' to be smaller than one during the estimation process. In some situation when the system is nonlinear and unknown, the estimates from the filter could have steady states as the system, such as flight vehicle, moves in the desired trajectory in [20]. On the location problem, several algorithms have been compared in the simulation of biopsy needle localization and it has been proved that the ROI-RK (Kalman filter based) algorithm is most useful because of ROI initialization step and EKF tracking step [21]. The advantages of the proposed method are illustrated as below

- 1)The proposed method is always feasible in the controllable system on the basis of mild assumptions.
- 2) It is practical in engineering because of low computing consumption as same as traditional EKF.
- 3)It provides much availability and potentiality for designing the controller of estimating process.

This note is organized as below. In Section II, the problem statement is presented. The estimation approach is presented in Section III. A numerical illustration of the proposed filter is given in Section IV, and concluding remarks are provided in Section V.

Throughout this technical note, we denote the  $n$ -dimensional unit matrix by  $I$  and we denote the spectral norm of matrices by  $\|\cdot\|$ .

## II. PROBLEM STATEMENT

Present a deterministic nonlinear system of discrete-time given by (1)

$$x_k = f(x_{k-1}) \quad (1a)$$

$$z_k = h(x_k) \quad (1b)$$

where  $x_k \in \mathbb{R}^n$  denotes the states of the system and  $z_k \in \mathbb{R}^m$

denotes the measurements taken from the system and  $k$  is the time instant. The  $f(\cdot)$  and  $h(\cdot)$  are both assumed to be  $C^1$  functions. For this system, we introduce an observer given by

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}) \quad (2a)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left( z_k - h(\hat{x}_{k|k-1}) \right) \quad (2b)$$

where the observer gain  $K_k$  is a time-varying  $n \times m$  matrix. The estimated states  $\hat{x}_{k|k-1}$  is named a priori estimate as well as  $\hat{x}_{k|k}$  a posteriori one. As  $f(\cdot)$  and  $h(\cdot)$  are assumed to be  $C^1$  functions, they are expanded via

$$f(x_k) - f(\hat{x}_k) = A_k (x_k - \hat{x}_k) + \phi(x_k, \hat{x}_k) \quad (3a)$$

$$h(x_k) - h(\hat{x}_{k|k-1}) = C_n (x_k - \hat{x}_{k|k-1}) + \nu(x_k, \hat{x}_{k|k-1}) \quad (3b)$$

where  $\phi$  and  $\nu$  are residuals and

$$A_k = \frac{\partial f}{\partial x}(\hat{x}_{k|k}) \quad (4a)$$

$$C_k = \frac{\partial h}{\partial x}(\hat{x}_{k|k-1}) \quad (4b)$$

We call  $A_k$  the ‘system matrix’. The states of system (1) are estimated according to the two-step EKF approach. The structure of the discrete-time EKF for system (1) is summarized in the steps as below.

Step 1: Assume  $k = 0$ , then define an initial condition

$$p(x_0/z^{-1}) = p(x_0) \quad \text{via} \quad \hat{x}_{0|0} = E[x_0] = \bar{x}_0$$

which is called the initial state estimation and

$$P_{0|0} = \text{cov}[x_0] = P_0 \quad \text{called the initial covariance}$$

matrix, where  $\text{cov}[x_0]$  denotes covariance function,  $-1$  means previous one sample time and  $-k$  means previous  $k$  sample time

Step 2: We can give the predictive mean as well as covariance matrix as below

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}) \quad (5a)$$

$$P_{k|k-1} = A_k P_{k-1|k-1} A_k^T + Q \quad (5b)$$

Step 3: The estimates of the states and covariance matrix are obtained by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left( z_k - h(\hat{x}_{k|k-1}) \right) \quad (6a)$$

$$P_{k|k} = P_{k|k-1} - K_k C_k P_{k|k-1} \quad (6b)$$

where

$$K_k = P_{k|k-1} C_k^T \left( C_k P_{k|k-1} C_k^T + R \right)^{-1} \quad (7)$$

Step 4: Let  $k = k + 1$ . Then the EKF algorithm always continues since Step 2.

Where  $Q^{n \times n}$  and  $R^{m \times m}$  are the symmetric positive definite matrices.

The estimation error is given by

$$\zeta_k = x_k - \hat{x}_{k|k-1} \quad (8)$$

We introduce the main result of exponential observer in [19] for illustration of the proposed case.

*Theorem 2.1:* For a nonlinear system given in equation (1) and an EKF as described in equation (2) to (7), the EKF can be regarded as an exponential observer if the following assumptions hold.

1) The inequalities given below are satisfied via

$$\|A_k\| \leq a \quad (9)$$

$$\|C_k\| \leq c \quad (10)$$

$$p_1 I \leq P_{k|k} \leq p_2 I \quad (11a)$$

$$p_1 I \leq P_{k|k-1} \leq p_2 I \quad (11b)$$

where  $a, c, p_1$  and  $p_2$  are positive real numbers.

2)  $A_k$  is nonsingular for every  $k \geq 0$ .

3) The nonlinear functions  $\phi(\cdot, \cdot)$  and  $\nu(\cdot, \cdot)$  in (3) are bounded via

$$\|\phi(x_k, \hat{x}_{k|k})\| \leq \kappa_\phi \|x_k - \hat{x}_{k|k}\|^2 \quad (12a)$$

$$\|\nu(x_k, \hat{x}_{k|k-1})\| \leq \kappa_\nu \|x_k - \hat{x}_{k|k-1}\|^2 \quad (12b)$$

for  $x_k, \hat{x}_{k|k-1}, \hat{x}_{k|k} \in \mathbb{R}^n$  with

$$\|x_k - \hat{x}_{k|k}\| \leq \varepsilon_\phi \quad (13a)$$

and

$$\|x_k - \hat{x}_{k|k-1}\| \leq \varepsilon_\nu \quad (13b)$$

respectively, where  $\varepsilon_\phi, \varepsilon_\nu, \kappa_\phi$  and  $\kappa_\nu$  are positive real numbers.

Using the direct method of Lyapunov, the extended Kalman filter is proved to be an exponential observer (the proving theory shown in [19]), i.e., the dynamics of the estimation error is exponentially stable under certain conditions 1), 2) and 3). Then we recall the definition of exponential observer as well as exponentially stable equilibrium point in [19]. Given  $\varepsilon, \eta$  and  $\theta$  are positive real numbers with  $\varepsilon > 0, \eta > 0$  and  $\theta > 1$ , the inequality given below is easily induced from (8)

$$\|\zeta_k\| = \|x_k - \hat{x}_{k|k-1}\| \leq \eta \|x_0 - \bar{x}_0\| \theta^{-k} \doteq \eta \|\zeta_0\| \theta^{-k} \quad (14)$$

for every  $k > 0$  where  $\zeta_0 \in D_\nu \doteq \{\nu \in \mathbb{R}^n \mid \|\nu\| < \varepsilon\}$ . It

follows from (14) that  $\zeta_k$  in (8) converges exponentially as below

$$\|\zeta_k\| = \eta \|\zeta_0\| \theta^{-k} \rightarrow 0 \text{ as } k \rightarrow \infty$$

It is easy to conclude that (14) guarantees the elimination of the estimation error in (8).

In general, to ensure such EKF being considered as an exponential observer, we conclude that certain restrictions are required as below.

i) The bounds (11a) and (11b) are checked online during the whole estimation process. They are naturally satisfied if the matrices  $A_k$  and  $C_k$  fulfill the uniform observability condition [19].

ii) The inequalities (12) is also checked online during the whole estimation process. The inequalities (12) depends on (13).

iii) A proper estimate of initial state is necessary. In fact, the real number  $\varepsilon$  in (14) is restricted complicatedly due to restrictions (11)~(13) ([19], Sec. II, Th. 7, p.2327). It is inclined to the small-scale convergence of the exponential observer.

Obviously, those restrictions probably keep EKF from a practical exponential observer.

The goal of this technical note is to propose a novel case that EKF becomes a practical exponential observer. And the estimates of the EKF converge quickly independent of the restrictions as remarked above.

### III. PROPOSED METHODS

We present the main proposition of Section III on the basis of the previous works. To make proving process concise, the direct method of Lyapunov theory is not shown in this paper as it can be found in [19]. We first prove the assumptions in theorem 3.1 satisfy all the conditions of theorem 2.1 through a detailed derivation and then the result is deduced directly from theorem 2.1.

*Theorem 3.1:* For a nonlinear system given in equation (1) and an EKF as described in equation (2) to (7), and assume the following assumptions hold.

- 1)  $f(\cdot)$  and  $h(\cdot)$  are both  $C^2$  functions.
- 2)  $\|A_k\|$  is nonsingular for every  $k \geq 0$ .
- 3) The inequalities given below are satisfied via

$$\|A_k\| < \bar{a} < 1 \quad (15a)$$

$$\|C_k\| < \bar{c} \quad (15b)$$

where  $\bar{a}$  and  $\bar{c}$  are positive real numbers.

Then the EKF can be an exponential observer independent of (11)~(13) for every bounded  $\zeta_0$ .

*Proof:* As  $h(\cdot)$  is a  $C^2$  function, applying the Taylor expansion yields

$$h(x_k) - h(\hat{x}_{k|k-1}) = C_k(x_k - \hat{x}_{k|k-1}) + \nu(x_k, \hat{x}_{k|k-1}) \quad (16)$$

where

$$\nu(x_k, \hat{x}_{k|k-1}) = \frac{h''(\rho)}{2} (x_k - \hat{x}_{k|k-1})^2 \quad (17)$$

denotes the remainder terms and  $h''(\cdot)$  is a second-order derivative with respect to  $x$  and  $\rho$  is a value between  $x_k$  and  $\hat{x}_{k|k-1}$ . There always exists a positive real number  $\kappa_\nu > 0$  such that

$$\left\| \frac{h''(\rho)}{2} \right\| \leq \kappa_\nu \quad (18)$$

From (17), we find that

$$\|\nu(x_k, \hat{x}_{k|k-1})\| \leq \kappa_\nu \|x_k - \hat{x}_{k|k-1}\|^2 \quad (19)$$

stands for any bounded  $x_k, \hat{x}_{k|k-1} \in \mathbb{R}^n$ .

Similarly, as  $f(\cdot)$  is a  $C^2$  function, there always exists a positive real number  $\kappa_\phi > 0$  such that

$$\|\phi(x_k, \hat{x}_{k|k})\| \leq \kappa_\phi \|x_k - \hat{x}_{k|k}\|^2 \quad (20)$$

stands for any bounded  $x_k, \hat{x}_{k|k} \in \mathbb{R}^n$ .

Considering the equations (5b), (6b) and (7), where  $Q$  and  $R$  are both symmetric positive definite matrixes. We obtain the results as below in terms of  $\|A_k\|$ .

- (1) When  $\|A_k\| \geq 1$ , from (5b), we can get that

$$P_{k|k-1} > 0, \text{ and } P_{k|k-1} \geq A_k P_{k-1|k-1} A_k^T \geq P_{k-1|k-1} \quad (21a)$$

Applying (6b) into (21a) and letting  $k \doteq k-1$  yields

$$P_{k|k-1} \geq P_{k-1|k-2} - K_{k-1} C_{k-1} P_{k-1|k-2} \geq (I - K_{k-1} C_{k-1}) P_{k-1|k-2} \quad (21b)$$

Obviously

$$K_{k-1} C_{k-1} > 0, \text{ and } 0 < I - K_{k-1} C_{k-1} < I \quad (22a)$$

Thus, from equation (21a) we get

$$P_{k|k-1} \geq P_{k-1|k-2} \quad (22b)$$

Similarly, from (6b) we can get

$$P_{k|k} = P_{k|k-1} - K_k C_k P_{k|k-1} = (I - K_k C_k) P_{k|k-1} \quad (23a)$$

Since (22a), applying (5b) into (23a) yields

$$P_{k|k} \geq A_k P_{k-1|k-1} A_k^T + Q \geq P_{k-1|k-1} \quad (23b)$$

We can infer that both  $P_{k|k-1}$  and  $P_{k|k}$  are monotonous

increase with no upper bound. The assumptions (11a) and (11b) are not satisfied. Thus when  $\|A_k\| \geq 1$ , the EKF is not able to be an exponential observer.

(2) In contrast, when  $\|A_k\| < 1$  we find that

$$0 < P_{k|k} \leq A_k P_{k-1|k-1} A_k^T + Q \leq A_1 P_{0|0} A_1^T + Q \quad (24)$$

Since  $Q$  is a symmetric positive definite matrix, it follows that there always exists a positive real number  $\bar{p}$  from (24) such that

$$0 < P_{k|k} \leq \bar{p}I \quad (25)$$

Similarly, from (5b), (6b), and (7) we can also get:

$$0 < P_{k|k-1} \leq A_k P_{k-1|k-2} A_k^T + Q \leq A_2 P_{1|0} A_2^T + Q \quad (26)$$

Like the above, there always exists a positive real number  $\bar{q}$  as well such that

$$0 < P_{k|k-1} \leq \bar{q}I \quad (27)$$

Applying (15), (19), (20), (25), and (27) to Theorem 2.1, we can conclude that the EKF can be regarded as an exponential observer independent of (11)–(13). That is for every bounded  $\zeta_0$ , the inequality (14) holds, that is the estimation error  $\zeta_k$  in (8) converges exponentially, independent of (11) to (13) only if positive real numbers  $\varepsilon > \|\zeta_0\|$ ,  $\eta > 0$  and  $\theta > 1$  holds  $\square$

From the above results, we can summary that the EKF of the nonlinear system can be a practical exponential observer since it is independent of those restrictions in Th.2.1 Section II. Instead of selecting the initial condition of the nonlinear system, any bounded counterpart is feasible for the observer. The assumptions in Th.3.1 are not restrictive. The estimates of the EKF converge quickly in the estimation process even without complicated restrictions. However, it should be noticed that the ‘system matrix’  $\|A_k\| < 1$  is the key.

#### IV. SIMULATION RESULTS

The proposed case is illustrated using an example of a range-only navigation system [22]. The range-only navigation system has the same form as systems (1). Besides that,  $f(\cdot)$  and  $h(\cdot)$  are  $C^2$  functions. The position of the platform called the autonomous underwater vehicle is estimated using only distances of measurements between the platform and the single beacon. Suppose the platform follows the continuous white noise motion model. For convenience, the navigation system is modified as below:

$$x_k = Ax_{k-1} \quad (28a)$$

$$y_k = \|x_k\| \quad (28b)$$

Where  $x_k \in \mathbb{R}^2$  denote the state of the system and  $y_k \in \mathbb{R}^1$  denote the output of the system.  $A$  is a  $2 \times 2$  matrix. EKF is used as an exponential observer.  $Q$  is the covariance matrix  $\Sigma_{w,k} = 1 \times I_2$  of Gaussian zero-mean state noise, and  $R$  is the variance  $\Sigma_{v,k} = 2^2$  of Gaussian zero-mean measurement noise. The examples of two different cases are given in terms of different values of the spectral norms of  $\|A\|$  as follows.

(1)  $\|A\| < 1$ . Let

$$A = \begin{bmatrix} \exp(-0.001T) & 0 \\ 0 & \exp(-0.002T) \end{bmatrix} \quad (29)$$

(2)  $\|A\| > 1$ . Let

$$A = \begin{bmatrix} \exp(0.001T) & 0 \\ 0 & \exp(0.002T) \end{bmatrix} \quad (30)$$

Where  $T=1$  denotes sample time for one second and  $\exp(\cdot)$  denotes the exponential function  $e^{(\cdot)}$ . The EKF filter is initialized with an initial position of [50 m, 50 m] and initial error covariance  $\begin{bmatrix} 10^{-8} & 0 \\ 0 & 10^{-8} \end{bmatrix}$ . To compared the estimation of two different cases, we use a root mean square error (RMSE) as below

$$RMSE = \|x - \hat{x}_{k|k}\| \quad (31)$$

There is no online check during the estimation. The estimation performance is demonstrated in Table I in terms of the initial position estimation error tolerance and RMSE, where  $\bar{x}_0$  and  $x_0$  denote the estimated and the true of the initial position respectively. The true coordinate of the initial state are (50,-50). The closest estimation of the initial states is (-60,60) while the furthest one is (-1000,1000). Their initial estimation errors are 155.6 and 1484.9 by the calculation according to (31). Their RMSE with both  $\|A\| < 1$  are only 0.045 and 1.377 respectively. By contrast, their RMSE with  $\|A\| > 1$  become large to  $10^7$ . Obviously, the proposed case gives much nicer results. Nevertheless, its computational cost is much small: only about 0.1 second with a 2.6 GHz CPU. Besides that, we can easily obtain the initial estimation error of (-500,500) is 777.8. Then the three percentages between each initial estimation error and its RMSE are 0.00009, 0.0008 and 0.0009. It is satisfactory that the RMSE is invisibly influenced by the estimation errors of the initial state.

TABLE I  
RESULTS OF TWO DIFFERENT CASES

$\bar{x}_0$ [m]	(-1000, 1000)	(-500, 500)	(-60, 60)
$x_0$ [m]	(50, -50)	(50, -50)	(50, -50)
RMSE with $\ A\  < 1$ [m]	1.377	0.620	0.045
RMSE with $\ A\  > 1$ [m]	$10^7$	$10^7$	$10^7$
time[sec]	0.114	0.114	0.113

The time response of the RMSE is depicted in Fig. 1 for several selected initial position estimations in Table I. As expected, the EKF with  $\|A\| < 1$  provides a much better estimation performance than the one with  $\|A\| > 1$ . The estimation of the former converges with time as shown in Fig.1(a). No matter how large the initial estimation error is, the estimation converges fast nearly in the same time. This shows the validity of the summary in Section III. By contrast, the EKF with  $\|A\| > 1$  becomes divergent rapidly as shown in Fig.1(b). The RMSE become so great after a moment of 300 seconds that the EKF is totally divergent. It is proved the proposed novel case with  $\|A\| < 1$  generally provides a high estimate quality with lower computational cost. By observing the estimation with  $\|A\| < 1$  and  $\|A\| > 1$ , it is possible that the convergence of the general EKF may reverse in practical application if there exists a little bit change of the system model in (28) as well as in (1).

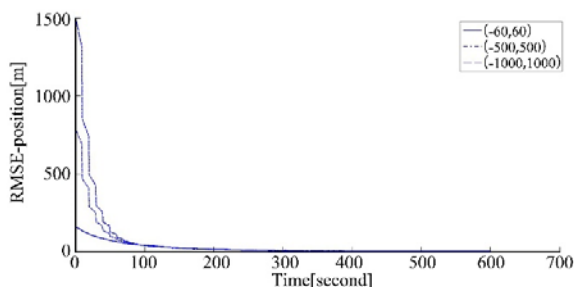


FIGURE 1(a). The convergence for the EKF filter with  $\|A\| < 1$

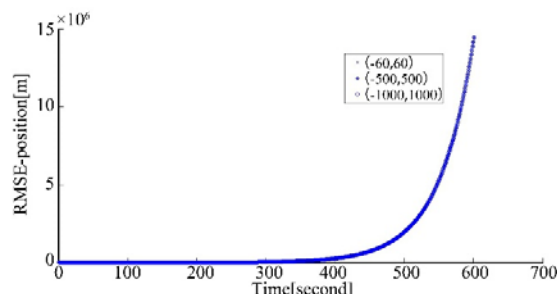


FIGURE 1 (b). The divergence for the EKF filter with  $\|A\| > 1$

## V. CONCLUSION

We proposed a novel case of practical exponential observer offering asymptotically exact estimation and computational efficiency. The observer consists of two parts: modified linearized system matrix and general EKF. The key technique is to set the spectral norm of the ‘system matrix’ smaller than one during the estimation process. The simulation experiments are given by an example of a range-only navigation system. Simulation results show that the proposed case provides a nice performance with little computation load even in the presence of poor initialization. Besides that, it is not necessary to have online check. Thus the proposed case can be a practical exponential observer. The theorem and the simulations demonstrate the convergence of the general EKF may become unpredictable in practical application if there exists the system model error. The novel case could be used in the state estimation of wireless communication networks while the security in broadcast is an important role and novel methods to improve the security in cooperative networks were proposed in [23–24]. And it could be in prospect in the navigation observer in unmanned surface vehicles [25] and closed-loop methods for the voltage-source inverters [26].

In this technical note, we give a supposed linearized system matrix. But it remains to design a desirable system matrix by regulating the inputs of the nonlinear systems. Possibly Model Prediction Control (MPC) can be used to design such a controller. In addition, we presume the deterministic nonlinear system in the paper and therefore the stochastic process of nonlinear system should be pursued in the proposed case as well as in the presence of unmeasurable states. It may refer to the event-triggered sliding mode controller for a class of uncertain stochastic systems addressed in [27] which provides a well control performance for a plant with unmeasurable state.

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