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INFRA-RED DIVERGENCES IN 2+1 DIMENSIONAL GAUGE THEORIES

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Abstract

Quantum field theories generally exhibit divergences. Ultra-violet divergences are treated through the renormalisation programme. Infra-red divergences, which accompany massless particles, are a characteristic of unbroken gauge theories and make it difficult to extract physical predictions. In this thesis we analyse various approaches to the infra-red problem and apply them to 2+1 dimensional gauge theories. These are useful as toy models, are related to the high temperature limit and are important in condensed matter physics. After briefly reviewing various responses to the infra-red problem in 3+1 dimensions, we begin our study of gauge theories in 2+1 dimensions by performing a one loop renormalisation of various on-shell Green's functions. Both the fermionic and scalar theories are employed to study the spin dependence of the infra-red structures. Ward identities are explicitly verified and gauge dependence is analysed by calculating in different gauges. Following arguments due to Kulish and Faddeev we see that the asymptotic interaction in QED cannot be neglected before or after scattering. This means that, even at asymptotic times, QED has a non-trivial gauge symmetry and so the Lagrangian fermion cannot be identified with a physical field. We then introduce a systematic method to construct locally gauge invariant dressed fields which describe particles moving with a well-defined velocity. We then find that the mass shift and the wave function renormalisation constants are infra-red finite when these dressed solutions are used. The infra-red structure of scattering is also analysed. Finally, the Bloch-Nordsieck method is used to study the IR problem at the level of the inclusive cross-section. It is seen that this method breaks down in 2+1 dimensions. Some suggestions for future work conclude this thesis.

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Chapter 1

Introduction

1.1 Field Theory and Infrared Divergences

The successful theories describing the fundamental forces (the electric, weak and strong interactions) are all based on gauge theories. The construction of these gauge theories requires that the particles transmitting the force be massless spin 1 bosons, known as gauge bosons. In Quantum Electrodynamics (QED), the gauge boson is the massless uncharged photon. It is an abelian theory described by the unitary group U(1). Quantum Chromodynamics (QCD) is a non-abelian gauge theory described by the SU(3) colour group. In QCD the gauge boson is the gluon [1,2]. The gluons are massless and carry colour charges. The weak interaction is also described by a gauge theory. However, the gauge symmetry is broken here and the gauge bosons

$W^{\pm}, Z^0$ are massive. It is well known in quantum field theories that calculations of physical quantities may have divergences in both the ultraviolet (UV) and the infra-red (IR) regions. In order to make sense of field theory, the problem raised by these divergences must be satisfactorily resolved. The renormalisation programme presented by 't Hooft [3] in 1971 provides a systematic response to the UV divergences in gauge theories. To solve the UV problem, we redefine our bare quantities in the Lagrangian density in terms of renormalised quantities.

On-shell renormalisation in unbroken gauge theories such as QED and QCD is, though, prevented by the appearance of infrared divergences. There are several responses to the IR problem in field theory calculations. Using a different renormalisation scheme is not a response, since the S-matrix elements will still have IR singularities. The most familiar answer is the Bloch-Nordsieck argument at the level of inclusive cross-sections [4–6] in QED. It is there argued that the experimental cross-section does not restrict the number of unobserved massless photons which may be emitted by any charged particle. This radical response means that QED does not have an S-matrix and is only suitably defined in terms of such inclusive cross-sections.

It is well understood that the physical origin of the the infrared problem is due to massless particles [6, 7]. In QED, the masslessness of photons allows them to travel over a large distance. This means that the potential between static charges falls off only as $1/r$ [8,9]. Kulish and Faddeev [10] were able to show that this implies that we can not switch off the coupling in the remote past and future in perturbation theory. The survival of these infinite range electromagnetic interactions led them to claim [10] that a good description of physical charged particles cannot be found. However, the Lagrangian fields can then never be directly associated with physical quantities, since in QED we have the following gauge transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \theta(x), \quad \text{and} \quad \psi(x) \to e^{ie\theta(x)}\psi(x). \tag{1.1}$$

If, as the infrared problem indicates, the coupling cannot be switched off, then the fermionic fields never become gauge invariant, and this is a minimal requirement for physical fields. We should therefore not expect to extract a description of physical charged particles from the gauge dependent Lagrangian fermion.

Finding a correct description of charged particles is then a highly non-trivial problem. In recent years, researchers at Plymouth, together with collaborators abroad, have developed a new approach to deal with this problem. To find a gauge invariant description of physical particles, we need to include a gauge bosonic cloud surrounding the charged particles. We call this a 'dressing'. This composite system corresponds to our charged particle. It must be both non-local and non-covariant [11–14]. In order to make our abelian matter field gauge invariant, we write,

$$\Psi(x) := h^{-1}(x)\psi(x) = e^{-ie\chi(x)}\psi(x) \tag{1.2}$$

where we demand that under a gauge transformation the dressing transforms as

$$h^{-1}(x) \rightarrow h^{-1}(x)e^{-ie\theta(x)}, \tag{1.3}$$

i.e.,

$$\chi(x) \rightarrow \chi(x) + \theta(x). \tag{1.4}$$

Generically we find that

$$\chi(x) = \frac{\mathcal{G} \cdot A}{\mathcal{G} \cdot \partial}. \tag{1.5}$$

where $\mathcal{G}^\mu$ is a first order differential operator and is constructed out of the vector ∂^μ, η^μ and v^μ that characterise the theory [7]. There is a great deal of freedom in solving this equation. To see which solution makes physical sense, first recall Dirac's description of a static charge [15, 16]. He noted that

$$\Psi(x) = \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right)\psi(x), \tag{1.6}$$

is gauge invariant and that it reproduces the electric field of a static charge. This can be easily shown using the equal time commutator,

$$[E_i(x), A_j(y)] = i\delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{1.7}$$

and the representation

$$\frac{\partial_i A_i}{\nabla^2}(x) = -\frac{1}{4\pi}\int d^3y \frac{\partial_i A_i(x_0, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{1.8}$$

So that the electric field acting on the state $\psi(x)|0\rangle$, produces the Coulombic field

$$E(x_0, \boldsymbol{y})\psi(x)|0\rangle = -\frac{e}{4\pi}\frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|^3}\psi(x)|0\rangle, \tag{1.9}$$

which is what we would expect for a static charge.

This corresponds to $\mathcal{G}^\mu = \eta^\mu \eta\cdot\partial - \partial^\mu$. For a particle moving with four velocity $u^\mu = \gamma(\eta, v)$ where η is the time-like unit vector and v is the velocity whose time component is zero, (1.6) can be generalised with $\mathcal{G}^\mu = (\eta + v)^\mu(\eta - v)\cdot\partial - \partial^\mu$. This can similarly be shown to yield the correct Liénard-Wiechert fields for a moving charge [13, 17]. It has been shown that with this new definition of fermionic fields, we can prove that physical quantities such as the on-shell wave function renormalisation constant are infrared finite in 3+1 dimensions [13]. As well as these good IR properties, the dressings have also led to a detailed understanding of the structure of the interactions between charges in QED and QCD. A large amount of this work was performed in 3+1 dimensions [7, 12, 17–29].

1.2 Field Theory in 2+1 Dimensions

We start this section by explaining why we should be interested in gauge theories in three dimensions when we live in a four dimensional universe and why the IR sector of these theories is so important. QED in 2+1 dimensions is a super-renormalisable theory as higher order terms in perturbation theory become less UV divergent [30,31]. As a result there are only a finite number of primitively divergent diagrams. This is due to the coupling constant e in 2+1 dimensional theories having dimensions $\sqrt{m}$. This simplification may, it is hoped, make such theories easier to solve. Feynman [31] used

2+1 dimensional theory as a *toy* model to study the non-perturbative confinement problem. A study of such theories may lead to some additional understanding about the properties of the four dimensional physical theories of the fundamental forces. QED in 2+1 dimensions can also be directly applied to condensed matter physics. This is because effective theories of condensed matter systems can be mapped into 2+1 dimensional QED [32–36]. The study of gauge theories in three dimensional space-time is also important due to their connection to the high temperature behaviour of four dimensional theories [37–41].

A significant problem in 2+1 dimensional QED is that perturbative calculations generate IR divergences, which are worse than those found in 3+1 dimensions.

By power counting, we can easily observe why such IR divergences occur in the on-shell Green's functions in four and three space-time dimensions. For instance in 3+1 dimensions the $\mathcal{O}(e^2)$ contribution to the three-point Green's functions involves the integral

$$\frac{1}{p^2 - m^2}\frac{1}{p'^2 - m^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{[(p-k)^2 - m^2][(p'-k)^2 - m^2]}. \tag{1.10}$$

Naive power counting tells us that this does not have an IR divergence when the outgoing particles are off-shell. However, if we extract a simple pole for each of the external legs, then the on-shell residue may be seen to be the IR divergent term

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{4p\cdot k\,p'\cdot k}, \tag{1.11}$$

where we have dropped higher powers of k which do not lead to IR divergences. If we consider (1.10) in 2+1 dimensions then we need to take more terms into account. We obtain

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{k^2} \frac{1}{4p\cdot k\, p'\cdot k} \left[1 + \frac{k^2}{2p\cdot k} + \frac{k^2}{2p'\cdot k}\right] , \tag{1.12}$$

which means that there are sub-leading terms which now become IR divergent. The purpose of this thesis is to investigate such IR properties using various techniques. As well as the approaches to the IR problem in 2+1 dimensions described in this thesis, we should also mention other techniques that have been applied to this area [30, 39, 40, 42–55].

1.3 Structure of the Thesis

The structure of the thesis is as follows. In Chapter 2, we consider QED in 3+1 dimensions. One loop calculations for 2-point and 3-point Green's functions are presented and we show how IR divergences occur in 3+1 dimensions. We then sketch an array of responses to the IR problem. A derivation of the asymptotic dynamics is presented which gives some insight into the physical origin of the IR divergences. We find that the asymptotic interaction Hamiltonian of QED cannot be neglected in 3+1 dimensions. This leads to the idea of constructing a dressed description of charged particles. We then discuss how these ideas have been tested in the perturbative domain and see that IR finite, on-shell Green's functions are yielded at all orders of perturbation theory.

In Chapter 3, we study gauge theories in 2+1 dimensions. Explicit calculations for the IR divergent on-shell renormalisation constants associated with the primitively divergent diagrams are presented. We consider both fermionic and scalar QED to study the spin dependence of these constants. Three different types of IR regulator are described. These are the introduction of a small photon mass, dimensional regularisation and a small off-shellness. We calculate the renormalisation constants in full, i.e., the leading and subleading divergences and the finite parts. Ward identities are explicitly verified and the gauge dependence is studied by calculating in different gauges.

Following the calculation of the various IR divergences which occur in the on-shell 2-point and 3-point Green's functions, we begin our study of the dressing method to deal with these IR divergences in 2+1 dimensions in Chapter 4. We find that the mass shift and the wave function renormalisation constants are IR finite when the solution of a systematic approach to constructing the dressing of a charge moving with a well-defined velocity is used [12]. A calculation of a charge scattering off a current is also presented and the Ward identity is investigated.

In Chapter 5, we consider the Bloch-Nordsieck method, which is the most common response to the IR problem at the level of the inclusive cross-section. We study both four and three dimensional QED. For simplicity, we choose to work in the scalar theory and consider scattering off different sources. It is seen that the method breaks down in 2+1 dimensions.

Finally, in Chapter 6 we conclude the thesis and make some suggestions for further studies.

Several appendices present various computational details at the end of the thesis. References are also given to the appendices throughout the text.

Chapter 2

Quantum Electrodynamics in 3+1 Dimensions

In this chapter we study Quantum Electrodynamics in 3+1 dimensions. In particular, we will study the IR divergences that occur in this theory and several responses to them. We begin with a brief discussion of perturbation theory and we will see how the IR divergences of on-shell Green's function occur in 3+1 dimensions. We will sketch the Bloch-Nordsieck approach to the cancellation of the IR singularities at the level of inclusive cross-sections. We then consider the formalism of Kulish and Faddeev [10] to discuss the physical origin of the IR problem. The authors of [10] have shown that the idea of switching off the coupling at large distances is no longer valid in an interacting field theory. They argued that even at asymptotically large times both before and after scattering the interaction Hamiltonian cannot be neglected. This implies that the fields do not become free. This will lead us to the framework introduced by the Plymouth group, which is based upon the idea of constructing the fields in such a way that they can be interpreted as charged particles. They have studied [7, 12, 14] the idea of dressing the matter field, i.e., surrounding it by the electromagnetic field which necessarily accompanies any charged particle (gluonic in the non-abelian theory). There have been various attempts to construct gauge invariant descriptions of charges [56–59]. There are two different ways to motivate the construction of the dressing used in this thesis. We can use arguments based on the work of Kulish and Faddeev, or we can use the heavy quark theory to develop the dressing equation. In this chapter we will discuss the former, and the latter can be found in Appendix C. We will discuss how these ideas have been tested in the perturbative domain where it was shown that IR finite, on-shell Green's functions are yielded at all orders of perturbation theory.

2.1 Perturbation Theory

In this section we will study the one loop superficially divergent Green's functions in 3+1 dimensional spinor electrodynamics. At this order, we have three primitively divergent diagrams: (i) the matter self-energy, (ii) the photon self-energy and (iii) the vertex correction diagram. Here we will only study the matter self-energy and the vertex correction diagrams, because there are no IR divergences in the photon self-energy diagram (as we will see later, this is a consequence of the photon being uncharged).

We begin by considering the propagation of an interacting matter field, in an on-shell renormalisation scheme. To proceed further, let us first recall (see for example Chapter 8 of [60]) the form of the Lagrangian density in an arbitrary Lorentz gauge with gauge parameter ξ. This is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2\,, \tag{2.1}$$

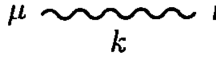
where ψ and $\bar{\psi}$ are the spinor field and its conjugate, A_μ is the gauge field and $D_\mu = \partial_\mu + ieA_\mu$ is the gauge covariant derivative. The field strength, $F_{\mu\nu}$, is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2.2}$$

and it is invariant under the gauge transformation of the field,

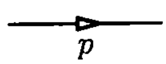
$$A_\mu \to A_\mu + \partial_\mu \Lambda\,. \tag{2.3}$$

In order to perform the perturbative calculations, we need the Feynman rules, which follow from the gauge fixed Lagrangian (see also Section 7.1 of [61]). We find that the photon propagator is



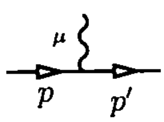
$$= \quad -iD_{\mu\nu}(k) \quad = \quad -\frac{i}{k^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right], \tag{2.4}$$

the fermion propagator is



$$= \quad iS(p) \quad = \quad \frac{i}{\not{p} - m}, \tag{2.5}$$

and finally, for the vertex we write



$$= \quad \Gamma^\mu \quad = \quad ie\gamma^\mu\,. \tag{2.6}$$

The diagrammatic representation of the expansion of the fermion two-point function up to one loop is shown in Figure 2.1. The first term in the expansion is the free field propagator.

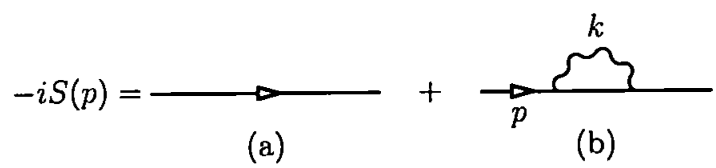


Figure 2.1: *Diagrammatic expansion of the matter propagator at one loop.*

Using these Feynman rules, we obtain the following expression for the diagrams of Figure 2.1.

$$\frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma(p))\frac{i}{\not{p} - m} \tag{2.7}$$

where the one loop fermion self-energy is given by

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}\gamma^\mu \frac{1}{\not{p} - \not{k} - m}\gamma^\nu . \tag{2.8}$$

To renormalise the matter propagator at one loop we introduce a counter-term diagram as shown in Figure 2.2. The Feynman rules for the counter-term diagram are given by

$$-i\Sigma^{\text{counter}} = \delta Z_2(\not{p} - m) + i\delta m . \tag{2.9}$$

where δm, the mass-shift, and δZ_2, the wave function renormalisation constants are introduced in the Lagrangian (2.1), by redefining

$$m \to m - \delta m \quad \text{and} \quad \psi_B \to \sqrt{Z_2}\psi_R , \tag{2.10}$$

with $Z_2 = 1 + \delta Z_2$.

Since we wish to perform on-shell renormalisation, the pole of the matter propagator must occur at $\not{p} = m$ and the residue of this pole must be equal to i. This will make the renormalised electron propagator look like a free field propagator at the physical mass (see, for example, Section 17.3 of [60]).

The renormalised electron propagator will be $i/(\not{p} - m - \Sigma^R(p))$, and so for the

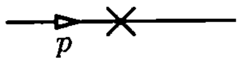


Figure 2.2: *Counter-term diagram for the matter propagator at one loop.*

pole to be at $p^2 = m^2$ when the particle is on-shell, we must have

$$\Sigma^R(p)\Big|_{\not{p}=m} = 0\,. \tag{2.11}$$

For the residue at $\not{p} = m$ to be i, we must have

$$\left.\frac{d\Sigma^R}{d\not{p}}\right|_{\not{p}=m} = 0\,. \tag{2.12}$$

We therefore have two renormalisation conditions

$$\begin{aligned} \delta m &= \Sigma(p)|_{\not{p}=m}\,, \\ \delta Z_2 &= \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m}\,, \end{aligned} \tag{2.13}$$

that determine the counterterms.

Before we can renormalise the theory, it is essential to be able to manipulate the divergences from the Feynman diagrams. There are several way we can regulate the divergent integrals. The simplest is to introduce a cut-off in the momentum integrals. Another is the Pauli-Villars regularisation in which a fictitious field with mass M is introduced. A discussion of Pauli-Villars regularisation can be found in Appendix B. Both of these methods are mainly used to regulate the UV divergences and they become problematic, in particular, when non-abelian gauge theories are concerned. In this chapter we will use dimensional regularisation. This is the most commonly used regularisation scheme in quantum field theory calculations. It has the crucial feature of preserving the gauge symmetry. The idea of dimensional regularisation is to generalise the action to an arbitrary dimension D, where there are regions in complex space in which the Feynman integrals are all finite. Then, as we analytically continue D to four dimensions, these divergences reveal themselves as poles in $1/(D-4)$ space, allowing us to absorb the divergences of the theory into the physical parameters.

Now we use dimensional regularisation to write down the expressions for the renormalisation constants associated with the fermion propagator in Feynman gauge. A calculation of the renormalisation constants can be found in many text books (see for example Chapter 9 of [61]). The expression for the mass shift (δm) is

$$\frac{\delta m}{m} = \frac{3e^2}{16\pi^2}\left[\frac{1}{\hat{\varepsilon}} + \ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{4}{3}\right] \tag{2.14}$$

where we have introduced an arbitrary mass scale, Λ, to redefine

$$e \longrightarrow e\Lambda^{\varepsilon}$$

and have defined $D = 4 - 2\varepsilon$. We have also made the standard definition

$$\frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon} - \gamma - \ln(4\pi)\,, \tag{2.15}$$

where γ is the Euler-Mascheroni constant. Notice that the divergence in the mass shift is an UV divergence and there are no IR infinities in the mass shift in 3+1 dimensions.

Similarly the expression for the wave function renormalisation constant, δZ_2, in Feynman gauge, using dimensional regularisation is

$$\delta Z_2 = -\frac{e^2}{16\pi^2}\left\{\left[\frac{1}{\hat{\varepsilon}_{\mathrm{UV}}} + \ln\left(\frac{\Lambda^2}{m^2}\right) + 4\right] + 2\left[\frac{1}{\hat{\varepsilon}_{\mathrm{IR}}} + \ln\left(\frac{\Lambda^2}{m^2}\right)\right]\right\} \tag{2.16}$$

The first square bracket is that which is needed to cancel the UV divergence for the fermion propagator. In addition, we now have an extra divergent term. This is due to the IR (small momentum) limit of the integral over the loop momentum, and this divergence occurs because the photon is massless. A discussion of this can also be found in many text books, in particular, in Chapter 17 of [60].

We can also use a small photon mass (μ) as an IR regulator, which we will discuss in Chapter 3. In this case we obtain

$$\delta Z_2 = -\frac{e^2}{16\pi^2}\left\{\left[\frac{1}{\hat{\varepsilon}_{\rm UV}} + \ln\left(\frac{\Lambda^2}{m^2}\right) + 4\right] + 2\ln\left(\frac{\mu^2}{m^2}\right)\right\} \tag{2.17}$$

Once again the first square bracket is due to the UV divergence of the propagator. The IR divergent part is now proportional to $\ln(\mu^2/m^2)$.

Next, we consider the three-point vertex diagram. Up to one loop, we have the diagrams of Figure 2.3. Using the Feynman rules, the sum of these diagrams is

$$\frac{i}{\not{p} - m}\gamma^\mu\frac{i}{\not{p} - m} + \frac{i}{\not{p}' - m}(ie\Gamma^\mu_{(1)})\frac{i}{\not{p} - m}, \tag{2.18}$$

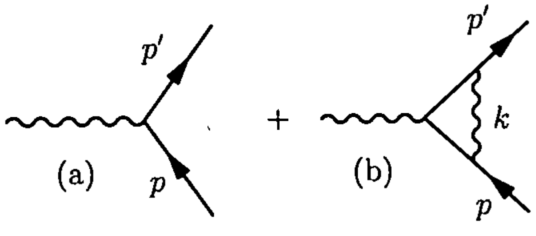


Figure 2.3: *One-loop three-point vertex diagrams.*

where the one loop vertex correction is given by

$$\Gamma^{\mu}_{(1)} = -i(ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^{\rho} \frac{i}{\not{p}' - \not{k} - m} \gamma^{\mu} \frac{i}{\not{p} - \not{k} - m} \gamma_{\sigma} D_{\rho\sigma}(k) . \tag{2.19}$$

To renormalise the three-point vertex at one loop we introduce the diagram of Figure 2.4. The Feynman rule for this vertex counter-term is

$$ie\gamma^{\mu}\delta Z_1 , \tag{2.20}$$

and we have the renormalisation condition

$$\delta Z_1 = \Gamma^{\mu R}_{(1)}(p = p') , \tag{2.21}$$

that determines the counter-term.

Note that $\Gamma^{\mu}_{(1)}$, the vertex correction, is a function of p and p'. If we take the limit of zero momentum transfer then the radiated photon carries no momentum and the vertex correction diagram 2.3(b) begins to look just like the self-energy diagram 2.1(b), i.e., $\Gamma^{\mu}_{(1)}$ is related to $\Sigma(p)$, as $p = p'$. This relation is called a Ward identity, and is formally obtained by realising that

$$\frac{i}{\not{p} - m} \gamma^{\mu} \frac{i}{\not{p} - m} = - \frac{\partial}{\partial p^{\mu}} \frac{i}{\not{p} - m} . \tag{2.22}$$

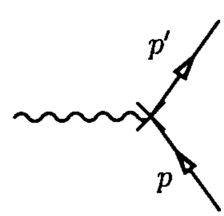


Figure 2.4: *Counter-term diagram for the three-point vertex .*

Using this in (2.8) and (2.19), we see that

$$\Gamma^{\mu}_{(1)}(p,p) = -\frac{\partial}{\partial p^{\mu}}\Sigma(p)\,. \tag{2.23}$$

Using the conditions (2.13) and (2.21) that determine the counterterms, it may also be written as

$$Z_1 = Z_2 \quad \text{or} \quad \delta Z_1 = \delta Z_2\,. \tag{2.24}$$

If we include the zero order contribution (2.22) becomes

$$\gamma^{\mu} + \Gamma^{\mu}_{(1)}(p,p) = \frac{\partial}{\partial p^{\mu}}(\not{p} - m - \Sigma(p))\,. \tag{2.25}$$

This is the Ward identity that relates the self-energy to the vertex correction at one loop, which can be generalised to all orders in perturbation theory.

An explicit calculation of δZ_1 in Feynman gauge using dimensional regularisation yields

$$\delta Z_1 = -\frac{e^2}{16\pi^2}\left\{\left[\frac{1}{\hat{\varepsilon}_{\mathrm{UV}}} + \ln\left(\frac{\Lambda^2}{m^2}\right) + 4\right] + 2\left[\frac{1}{\hat{\varepsilon}_{\mathrm{IR}}} + \ln\left(\frac{\Lambda^2}{m^2}\right)\right]\right\}\,. \tag{2.26}$$

As expected we recover $\delta Z_1 = \delta Z_2$. This shows that dimensional regularisation preserves the Ward identity (which is to be expected since it preserves gauge invariance).

We will finish this section by briefly describing the idea of the Bloch-Nordsieck approach to cancel the IR infinities. This was first presented by Bloch and Nordsieck in [4], and was written long before the invention of relativistic perturbation theory. Here we will follow a modern, and simplified, version of the analysis due to Weinberg. (See [6] or Chapter 13 of [62]).

In an experimental measurement any detector will have a finite resolution. Since there is no lower bound on the energy of a real photon, $\omega(k) = |\mathbf{k}|$, an electron can be accompanied by any number of soft photons as long as their total energy is less than the resolution of the detector. The idea behind the Bloch-Nordsieck method is that the IR divergences arising in QED can be removed at the level of the inclusive cross-section by adding the sum of all the real soft photon emissions. We have to calculate the various cross-sections for the emission of zero, one, two (and so on) real photons separately and then add all of these cross-sections to get the experimental result. This is found to be finite in 3+1 dimensions. This also explain why a small photon mass (μ) is an IR regulator: a detector with resolution less than μ could distinguish between an electron and an electron accompanied by a soft photon. A more detailed discussion of the Bloch-Nordsieck method in 3+1 dimensions and new calculations in 2+1 dimensions will be presented in Chapter 5.

2.2 Asymptotic Dynamics and IR Divergences in QED

In perturbation theory it is generally assumed that the coupling switches off at large times. This is because, at large times, the particles are widely separated and behave like free particles. This assumption, which is the basis of the LSZ formalism, is incorrect in the theory, where the incoming and outgoing systems include bound states (e.g., confined quarks). It is also incorrect if the physics is characterised by long range interactions which can be generated by massless gauge bosons.

The most obvious candidate for the second scenario is QED, which has a long range interaction. The masslessness of the photon means that the potential between static charges falls off only as $1/r$ in 3+1 dimensions. It has been known for a long time [8,9] that this means that switching off the coupling at the remote past and distant future generates IR divergences in the wave function renormalisation constant, which we have encountered in the previous section.

This has been studied [10] in the relativistic theory by Kulish and Faddeev (KF). In recent years the Plymouth group has been studying [7, 24, 25] this approach to produce a better description of charged particles [12], which we shall discuss in the next section. We shall now give a brief discussion of the work of KF and what their results mean for QED.

We start from the usual interaction Hamiltonian for QED [25]

$$\mathcal{H}_{\text{int}}(t) = -e \int d^3x \, A_\mu(t,x) J^\mu(t,x) \,, \tag{2.27}$$

where $J^\mu(t,x) = \overline{\psi}(t,x)\gamma^\mu\psi(t,x)$ is the conserved matter current. In order to calculate the LSZ reduction formula, which relates the S-matrix to the Green's functions of the field theory, we must be in the interaction picture. Although the time evolution of the states is determined by (2.27), we assume that the evolution of the fields is asymptotically given by the free Hamiltonian [25]. Then we can insert the free field expansion in (2.27). The plane wave expansions in terms of particle creation and annihilation operators, are chosen to be

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ b(p,s)u^s(p)e^{-ip\cdot x} + d^\dagger(p,s)v^s(p)e^{ip\cdot x} \right\} , \tag{2.28}$$

$$\overline{\psi}(x) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \left\{ d(q,r)\overline{v}^r(q)e^{-iq\cdot x} + b^\dagger(q,r)\overline{u}^r(q)e^{iq\cdot x} \right\} , \tag{2.29}$$

and

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \left\{ a_\mu(k)e^{-ik\cdot x} + a_\mu^\dagger(k)e^{ik\cdot x} \right\} , \tag{2.30}$$

where $E_p = \sqrt{|p|^2 + m^2}$ and $\omega_k = |k|$ are the usual energy terms.

Following KF we now substitute these expansions into the interaction Hamiltonian. This results in eight terms. Integrating out the spatial variable x, we obtain a momentum delta-function which we use to perform the q integration. The resulting integrals will now involve only p and k integrations. The time dependence is then of the form $e^{i\Phi t}$, where Φ is made up of sums and differences of the energy terms.

We first consider sums of energy terms:

$$\Phi = E_{p+k} + E_p + \omega_k \, . \tag{2.31}$$

There are two such terms, coming from the "off-diagonal" terms $v\gamma^\mu\overline{u}a_\mu^\dagger$ and $u\gamma^\mu\overline{v}a_\mu$ and there are no values of p and k for which these Φ's are zero. KF claimed that these part of the asymptotic Hamiltonian will therefore vanish, because in (2.31) $e^{i\Phi t}$ will oscillate rapidly as $t \to \pm\infty$.

Next, we shall consider the Φ's of the form

$$\Phi = E_{k-p} + E_p - \omega_k \,. \tag{2.32}$$

There are two such terms, coming from $\overline{v}\gamma^\mu u a_\mu^\dagger$ and $\overline{u}\gamma^\mu v a_\mu$. In order that (2.32) vanish as for large t, we require

$$\omega_k = E_{k-p} + E_p \Rightarrow \boldsymbol{k}^2 = \boldsymbol{p}^2 + m^2 + (\boldsymbol{k}-\boldsymbol{p})^2 + m^2 \,. \tag{2.33}$$

This only has a solution if $m_\gamma^2 > 4m^2$, and as we have $m_\gamma = 0 \neq m$, this equation has no solution. We can use the above arguments to argue with KF that the Hamiltonian for these parts also vanishes for large t.

We are left with four "diagonal" terms. Let us consider the diagonal term of the form

$$-e\int \frac{d^3x\, d^3p\, d^3q\, d^3k}{(2\pi)^9)} \frac{a_\mu(k)}{2\omega_k\sqrt{4E_pE_q}}\, b^\dagger(q,r)\, b(p,s)\, \overline{u}^r(q)\, \gamma^\mu\, u^s(p) e^{iq\cdot x} e^{-ip\cdot x} e^{-ik\cdot x} \,. \tag{2.34}$$

Integrating out the $\boldsymbol{x}$ integral gives a delta function of the form $\delta^3(\boldsymbol{q}-\boldsymbol{p}-\boldsymbol{k})$. Integrating out the $\boldsymbol{q}$ integral yields

$$-e\int \frac{d^3p\, d^3k}{(2\pi)^6)} \frac{a_\mu(k)}{2\omega_k\sqrt{4E_pE_{p+k}}}\, b^\dagger(p+k,r)\, b(p,s)\, \overline{u}^r(p+k)\, \gamma^\mu\, u^s(p) e^{i\Phi t} \,, \tag{2.35}$$

where

$$\Phi = E_{p+k} - E_p - \omega_k \,. \tag{2.36}$$

For this to vanish we require

$$E_{p+k} = E_p + \omega_k \Rightarrow \boldsymbol{p}\cdot\boldsymbol{k} = |\boldsymbol{k}|\, \sqrt{|\boldsymbol{p}|^2 + m^2} \,. \tag{2.37}$$

This structure is non-zero if $\boldsymbol{k} = 0$, which is where the IR problem arises. KF argue that as this Φ vanishes $e^{i\Phi t}$ will not oscillate to zero and so the interaction Hamiltonian cannot be neglected. (This is only solved for $|\boldsymbol{k}| = 0$, because the photon is massless.) For our choice of normalisation $\overline{u}^r(p)\gamma^\mu u^s(p) = 2p^\mu\delta^{rs}$ (see [7]), the interaction Hamiltonian $\mathcal{H}_{\rm int}$ corresponding to this is

$$-e\int \frac{d^3p\,d^3k}{(2\pi)^6}\,\frac{p^\mu}{2\omega_k E_p}\,b^\dagger(p,r)\,b(p,r)\,a_\mu(k)\,e^{-it(E_p-E_{p+k}+\omega_k)}\,. \tag{2.38}$$

where we have dropped $\boldsymbol{k}$ in the non-sensitive places. The exponent in (2.38) can also be written as

$$E_p - E_{p+k} + \omega_k \;=\; E_p - E_p\sqrt{1+\frac{2\boldsymbol{p}\cdot\boldsymbol{k}}{E_p^2}+\mathcal{O}(k^2)}+\omega_k \;=\; -\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{E_p}+\omega_k+\mathcal{O}(k^2)\,. \tag{2.39}$$

In the limit $k^2 \longrightarrow 0$ the expression (2.38) may be written as

$$-e\int \frac{d^3p\,d^3k\,d^3y}{(2\pi)^6}\,\frac{p^\mu}{E_p}\,b^\dagger(p,r)\,b(p,r)\,A_\mu^{(+)}(y)\,e^{-i\boldsymbol{k}\cdot\left(\boldsymbol{y}-\frac{t}{E_p}\boldsymbol{p}\right)}\,, \tag{2.40}$$

where we have used the identity

$$\int d^3y\,e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\,A_\mu^{(+)}(y) = \frac{1}{2\omega_k}a_\mu(k)e^{-i\omega_k t}\,. \tag{2.41}$$

After integrating out the $\boldsymbol{k}$ integral this part of the interacting Hamiltonian yields

$$-e\int \frac{d^3p\,d^3y}{(2\pi)^3}\,\frac{p^\mu}{E_p}\,b^\dagger(p,r)\,b(p,r)\,A_\mu^{(+)}(k)\,\delta^3\left(\boldsymbol{y}-\frac{t}{E_p}\boldsymbol{p}\right)\,. \tag{2.42}$$

The three other "diagonal" terms can be similarly evaluated and we obtain the following final form of the asymptotic interaction Hamiltonian for the massive particle.

$$\mathcal{H}_{\rm int}^{\rm as} = \int d^3y\,A_\mu(t,y)J_{\rm as}^\mu(t,y)\,, \tag{2.43}$$

where the asymptotic current is defined by

$$J^{\mu}_{\rm as}(t,y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{p^{\mu}}{E_p}[b^{\dagger}(p,r)\,b(p,r) - d^{\dagger}(p,r)\,d(p,r)]\,\delta^3\left(y - \frac{t}{E_p}\mathbf{p}\right) . \tag{2.44}$$

Here we have used normal ordering to replace $d\,d^{\dagger}$ by $d^{\dagger}\,d$ and note that the square bracket in (2.44) is just the charge density $\rho(p)$. $J^{\mu}_{\rm as}$ is thus the current associated with a charged particle moving with velocity p^{μ}/E_p. This observation implies that *the asymptotic dynamics of QED is not that of a free theory but is closely related to the classical theory.*

It is this that underlies KF's dramatic statement [10] that: "the relativistic concept of a charged particle does not exist". This discussion of asymptotic dynamics can be extended to scalar QED, and may be found in [25]. This shows that the asymptotic dynamics in abelian gauge theories are spin independent and exactly the same asymptotic interaction (2.43) is found.

We now ask if KF's conclusion, that there are no charged particles, is really necessary.

2.3 Construction of Charges

The survival of the asymptotic interaction in QED means that we cannot set the electromagnetic coupling to zero for the incoming and outgoing fields. As a consequence of this we see that the matter field, $\psi(x)$, is not gauge invariant in the remote past and distant future, and therefore it cannot be viewed as a physical field. To construct a gauge invariant charged field, we define

$$\Psi(x) = h^{-1}(x)\psi(x)\,, \tag{2.45}$$

where $h^{-1}(x)$ is a functional of the fields, which we call the dressing [7, 12]. Under a local gauge transformation

$$\psi(x) \to e^{ie\theta(x)}\psi(x)\,, \tag{2.46}$$

we demand

$$h^{-1}(x) \to h^{-1}(x)e^{-ie\theta(x)}\,, \tag{2.47}$$

i.e., a dressed matter field is gauge invariant and can potentially be identified with a physical particle. The form of this gauge dependent quantity, $h^{-1}(x)$, needs to be made more precise.

In order to construct the dressing, we will use arguments based on the form of the asymptotic interaction Hamiltonian. We start by writing the interacting Hamiltonian as follows:

$$\mathcal{H}^{\text{as}}_{\text{int}}(t) = -e\int d^3x\, A^h_\mu(t,x)J^\mu(t,x)\,, \tag{2.48}$$

where

$$A^h_\mu = A_\mu + \frac{1}{ie}\partial_\mu(h^{-1})h\,, \tag{2.49}$$

which we recognise as a (field dependent) gauge transformation of the vector potential. From (2.47) we see that h can be interpreted as a dressing. Written in terms of the dressed (charged) field, the asymptotic interaction Hamiltonian becomes

$$\begin{aligned} \mathcal{H}^{\text{as}}_{\text{int}} &= \int d^3x\, A^h_\mu(t,x) J^\mu_{\text{as}}(t,x) \\ &= \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E_p} A^h_\mu(t,x)\rho(p)\delta^3\left(x - \frac{p}{E_p}t\right), \end{aligned} \tag{2.50}$$

where

$$\rho(p) = J^0_{\text{as}} = b^\dagger(p,r)\, b(p,r) - d^\dagger(p,r)\, d(p,r), \tag{2.51}$$

is the charge density. This asymptotic Hamiltonian will vanish if we can construct a dressing such that $A^h_\mu(t,x)p^\mu = 0$, for p^μ an on-shell four vector. We cannot solve this simultaneously for all p^μ, but we solution will exist at any one point on the mass shell [7]. Thus, at the point where $p^\mu = mu^\mu$, the dressing must satisfy

$$u^\mu A^h_\mu(t,x) = 0, \tag{2.52}$$

which we call the *dressing equation* [12]. This equation together with the fundamental requirement of the gauge invariance of the dressed field $h^{-1}\psi$, will determine the form of the dressing. Such a dressed field will have free asymptotic dynamics at the correct point on the mass shell.

From (2.49), the dressing equation (2.52) can also be written as

$$u \cdot \partial h^{-1}(x) = -ieh^{-1}(x) u \cdot A(x), \tag{2.53}$$

with $u^\mu = \gamma(\eta + v)^\mu$, where η^μ is the time like unit vector $(1, 0)$, v^μ is the space like vector $(0, \boldsymbol{v})$ with $\boldsymbol{v}$ the three velocity of the charged particle and $\gamma = (1 - \boldsymbol{v}^2)^{-1/2}$ the standard relativistic factor.

For simplicity we show how to satisfy these requirements for the case of a static charge $u^\mu = (1,0,0,0)$. In this case the dressing equation (2.53) becomes

$$\partial_0 h^{-1}(x) = -ieh^{-1}(x)\, A_0(x)\,. \tag{2.54}$$

To proceed further, let $h^{-1} = \exp(-ie\partial_i A_i/\nabla^2)$, which we recall was Dirac's proposal [16]. After using the identity

$$\partial_\mu e^{\mathrm{O}} = e^{\mathrm{O}}\left(\partial_\mu \mathrm{O} + \frac{1}{2}[\partial_\mu \mathrm{O}, \mathrm{O}]\right)\,, \tag{2.55}$$

where O is an arbitrary operator whose commutator $[\partial_\mu \mathrm{O}, \mathrm{O}]$ is a c-number, we find that

$$\begin{aligned} \partial_0 h^{-1}(x) &= \partial_0 \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right) \\ &= -ie\exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right)\left(\frac{\partial_0\partial_j A_j}{\nabla^2}(x) + \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^2}\right)\,. \end{aligned} \tag{2.56}$$

From this we see that Dirac's proposed description (1.6) does not satisfy (2.54), even if we ignore the e^2 term.

It is well known that the solution to equations like (2.54) generally have the form [12]

$$h^{-1}(x,a) \approx \tilde{T}\exp\left(-ie\int_a^t A_0(s,\mathrm{x})ds\right)\,, \tag{2.57}$$

where we have introduced an arbitrary time a which asymptotically has no physical significance. This clearly satisfies the dressing equation (2.54), but it does not satisfy the gauge transformation property (2.47) which is also fundamental for a dressing.

Under a gauge transformation we have

$$h^{-1}(x,a) \to e^{ie\theta(a,x)}\, h^{-1}(x,a)\, e^{-ie\theta(x)}\,. \tag{2.58}$$

In order to satisfy the gauge transformation property, we seek a solution of the form

$$h^{-1}(x,a) = \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(a,x)\right) \tilde{T}\exp\left(-ie\int_a^t A_0(s,x)ds\right)\,. \tag{2.59}$$

We can write this as

$$h^{-1}(x,a) = \tilde{T}\exp\left(-ie\int_a^t \left[A_0(s,x) - \frac{\partial_0\partial_i A_i}{\nabla^2}(s,x)\right] ds\right) \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right)\,, \tag{2.60}$$

where we have combined the a dependent terms under one exponential. It can also be written as

$$\begin{aligned} h^{-1}(x,a) &= \tilde{T}\exp\left(-ie\int_a^t \left[\frac{\partial^i\partial_i A_0}{\nabla^2}(s,x) - \frac{\partial_0\partial_i A_i}{\nabla^2}(s,x)\right] ds\right) \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right) \\ &= \tilde{T}\exp\left(-ie\int_a^t \frac{\partial^i F_{i0}}{\nabla^2}(s,x)ds\right) \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right)\,. \end{aligned} \tag{2.61}$$

It is easy to see that this expression for the solution of the dressing equation satisfies the gauge transformation property (2.47). Under a gauge transformation $A_\mu(x) \to A_\mu(x)+\partial_\mu\theta(x)$, the first exponential is gauge invariant while the second one transforms as

$$\exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right) \to \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}(x)\right) \exp(-ie\theta(x))\,. \tag{2.62}$$

Thus we obtain a factorisation of the static dressing into two parts in (2.61). The second part in (2.61) is a *minimal* dressing, which is essential for gauge invariance and can also be recognised as Dirac's original proposal for the static dressing. The first factor in (2.61) is an additional dressing, which is separately gauge invariant and was missed in Dirac's proposal. Indeed the electromagnetic field associated with the static dressing (2.61) is the same as the Dirac minimal part, since the additional part will commute with the electric and magnetic field.

It can similarly be shown [12] that the dressing needed to describe a charged particle moving with velocity $u^\mu = (\eta, v)^\mu$ is

$$h^{-1}(x) = e^{-ieK(x)}e^{-ie\chi(x)}\,, \tag{2.63}$$

where

$$\chi(x) = \frac{\mathcal{G}\cdot A}{\mathcal{G}\cdot\partial}\,, \tag{2.64}$$

is the minimal part of the dressing with $\mathcal{G}^\mu = (\eta+v)^\mu(\eta-v)\cdot\partial-\partial^\mu$, and the additional part of the dressing is

$$K(x) = \int_{\pm\infty}^{x_0}(\eta+v)^\mu\frac{\partial^\nu F_{\mu\nu}}{\mathcal{G}\cdot\partial}(x_s)ds\,. \tag{2.65}$$

The full derivation of this result can be found in [12] including a detailed treatment of the limit of integration a. It is straightforward to see that (2.64) and (2.65) reproduce (2.61) at the static limit (up to the limit of integration).

As in the static case the additional part does not affect the electromagnetic configuration. It was shown in [12] that this dressing generates the correct electric and magnetic fields. We now proceed to describe how these dressed fields have been tested in 3+1 dimensions.

2.4 Perturbation Theory with Dressed Fields

In this section we will study the effect of dressings upon the IR divergences associated with on-shell Green's functions at one loop. To do this we sketch the calculations performed in [7, 23]. We first show that a dressed charge propagating with a physical mass is free of IR divergences if the dressing is appropriate for the point on the mass shell where we renormalise. Then we study a charge scattered off a source and show that this is also free of IR divergences. Following [23], we only consider the IR structures in loop integrals and drop structures which are IR finite. It is important to realise that these IR structures are gauge invariant, since the dressed charged fields are gauge invariant by construction.

In order to perform perturbative calculations, we first need to know the Feynman rules. In a perturbative expansion of the Green's functions of dressed fields, as well as including the usual interaction vertices, we must also expand the dressings, since they explicitly depend on the coupling, e. As a consequence, we introduce new vertices and hence new diagrams. The dressing provides two different vertex structures from each of the two factors. The Feynman rules for the dressed Green's functions are the usual ones described in Section 2.1 with the addition of two new rules corresponding to the dressings as shown in Figure 2.5.

We have here defined V and W as follows [7]:

$$V^{\mu} := k \cdot (\eta - v)\,(\eta + v)^{\mu} - k^{\mu}, \quad W^{\mu} = \frac{k \cdot (\eta - v)\, k^{\mu} - k^{2}\,(\eta + v)^{\mu}}{k \cdot \eta}, \tag{2.66}$$

where $v = (0, \boldsymbol{v})$ is the velocity of the particle with momentum $p = m\gamma(1, \boldsymbol{v})$ and k is the incoming momentum of the photon.

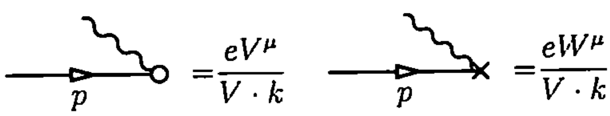


Figure 2.5: *The Feynman rules from expanding the dressing. The first vertex comes from the minimal (χ) part of the dressing, and the latter corresponds to the additional dressing, the K term.*

Electron propagator

Let us first consider the electron propagator in fermionic QED. At one loop we have, as well as the usual covariant diagram, contributions from the perturbative expansion of the dressing. The relevant diagrams are shown in Figure 2.6. For simplicity we only include the minimal part (χ term) of the dressing as it has been shown that the additional part does not introduce soft IR divergences in the 3+1 dimensional propagator [7].

Now we use the Feynman rules to write down the expression for each diagrams. The contribution of the usual covariant diagram 2.6(b) is

$$\frac{e^{2}}{\not{p} - m} \int \frac{d^{4}k}{(2\pi)^{4}}\, \gamma^{\mu} \frac{1}{\not{p} - \not{k} - m} \gamma^{\nu} D_{\mu\nu}(k) \frac{1}{\not{p} - m}, \tag{2.67}$$

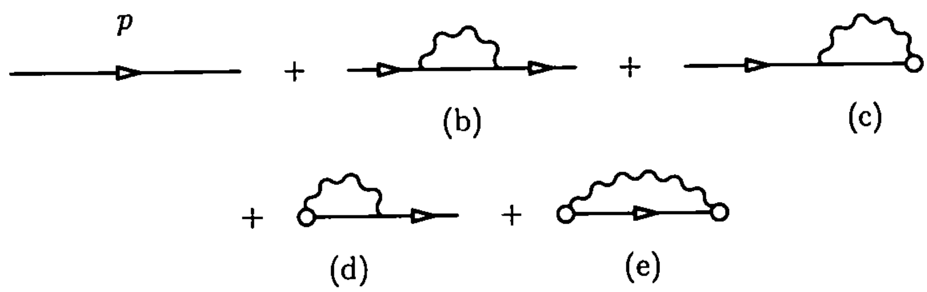


Figure 2.6: *All one-loop Feynman diagrams in the electron propagator which contain IR-divergences when we include the minimal dressing.*

where the form of the photon propagator is left completely general to highlight the gauge invariance of our final result. This diagram has a double pole and is well defined, since it is IR finite if p^μ is off-shell. This double pole will be killed off via mass renormalisation. However, as we have seen in Section 2.1 the wave function renormalisation generates IR divergences which can be found by extracting a pole in (2.67). To establish a single pole structure we rewrite this as

$$\frac{e^2}{\not p - m} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{\not p - \not k + m}{(p-k)^2 - m^2} \gamma^\nu D_{\mu\nu}(k) \frac{1}{\not p - m} \,. \tag{2.68}$$

We now use

$$(\not p + m)\gamma^\nu = 2p^\nu - \gamma^\nu(\not p - m) \tag{2.69}$$

and obtain

$$\frac{e^2}{\not p - m} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{2p^\nu}{(p-k)^2 - m^2} D_{\mu\nu}(k) \frac{1}{\not p - m} \,, \tag{2.70}$$

where we have dropped the single pole term and the term proportional to k in the numerator, as they are IR finite. Next, we expand the integral in $(p^2 - m^2)$ using the formula

$$I(p^2 - m^2) = I_0(p^2 = m^2) + (p^2 - m^2)I_1(p^2 = m^2) + \mathcal{O}((p^2 - m^2)^2)\,, \qquad (2.71)$$

with

$$I_1 = \frac{p^\mu}{2m^2}\frac{\partial I}{\partial p^\mu}\,. \qquad (2.72)$$

We obtain

$$-\frac{e^2}{\not{p} - m}\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{2p^\nu(p^2 - m^2)}{(2p\cdot k)^2}D_{\mu\nu}(k)\frac{1}{\not{p} - m}\,. \qquad (2.73)$$

We now write $(p^2 - m^2) = (\not{p} + m)(\not{p} - m)$, and the $(\not{p} - m)$ factor will remove the double pole while the $(\not{p} + m)$ factor may be taken through the remaining gamma matrix to regain $2p^\mu$ plus a term which does not contain a pole in the propagator. Using this (2.73) becomes

$$-\frac{e^2}{\not{p} - m}\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{p^\mu p^\nu}{(p\cdot k)^2}D_{\mu\nu}(k)\,. \qquad (2.74)$$

Naive power counting shows that this has an IR divergence which corresponds to the IR divergence in wave function renormalisation.

Next we consider the diagram 2.6(c). The contribution of this diagram to the propagator is

$$e^2\int \frac{d^4k}{(2\pi)^4}\frac{V^\mu}{V\cdot k}\frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2}\gamma^\nu \frac{1}{\not{p} - m}D_{\mu\nu}(k)\,. \qquad (2.75)$$

Again we use (2.69) and drop the IR finite term to obtain

$$\frac{e^2}{\not{p} - m}\int \frac{d^4k}{(2\pi)^4}\frac{V^\mu p^\nu}{V\cdot k p\cdot k}D_{\mu\nu}(k)\,. \qquad (2.76)$$

The contribution of diagram 2.6(d) is easily seen to be identical to this.

Finally, we need to find the contribution of diagram 2.6(e). We will refer to this class of diagrams as rainbow diagrams. From the Feynman rules we obtain

$$e^2 \int \frac{d^4k}{(2\pi)^4} \frac{V^\mu V^\nu}{(V\cdot k)^2} \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} D_{\mu\nu}(k)\,. \tag{2.77}$$

It is easy to see that this diagram is not well defined since it IR divergent even when p^μ is off-shell. In order to make sense of this diagram, we follow [23] (a detailed discussion of this will also be given in Chapter 4) and obtain

$$-\frac{e^2}{\not{p} - m} \int \frac{d^4k}{(2\pi)^4} \frac{V^\mu V^\nu}{(V\cdot k)^2} D_{\mu\nu}(k)\,. \tag{2.78}$$

We now combine these results to obtain the following form for the IR divergent term in the single pole, for the case of the electron propagator, i.e.

$$-\frac{e^2}{\not{p} - m} \int \frac{d^4k}{(2\pi)^4} \left[\frac{p^\mu}{p\cdot k} - \frac{V^\mu}{V\cdot k}\right] D_{\mu\nu}(k) \left[\frac{p^\nu}{p\cdot k} - \frac{V^\nu}{V\cdot k}\right]. \tag{2.79}$$

This term is gauge invariant, because any modification of the Feynman gauge photon propagator will involve either a k^μ or k^ν factor, and these extra structures will vanish on multiplying into the square bracket in the above structure.

There is a simple argument in [23] to show that the sum of the different IR divergences vanishes completely, which uses the following identity:

$$\frac{V^\mu}{V\cdot k} = \frac{(\eta+v)^\mu(\eta-v)\cdot k - k^\mu}{(k\cdot\eta)^2 - k^2 - (k\cdot v)^2} = \frac{(\eta+v)^\mu(\eta-v)\cdot k}{(k\cdot\eta)^2 - (k\cdot v)^2} = \frac{p^\mu}{p\cdot k}\,. \tag{2.80}$$

The term proportional to k^μ can be dropped, using the argument for the gauge invariance of (2.79). We have also neglected k^2 in the denominator, as it vanishes for soft divergences. This argument *only* holds if we renormalise at the correct point on the mass shell, i.e., $p = m\gamma(\eta + v)$. We thus see that the dressed two point function does not suffer from IR divergences and that the Green's function has the simple pole characteristic of particle propagation.

We further note that the cancellation of the IR divergences that occur in the electron propagator in 3+1 dimensions can be seen to be spin independent [7, 23]. This completes our study of IR divergences in the dressed matter propagator, and we now move on to the case of a charge being scattered off a source.

Scattering charges

Consider a charge scattered off a current [7]. The relevant diagrams at one loop are shown in Figure 2.7 where we only display those diagrams which can generate an IR divergence and a pole for each of the external legs. Diagrams 2.7(b)– 2.7(e) are just propagator corrections on one or other legs and as such have been calculated above in the propagator. The only fresh diagrams that we need to caculate here are 2.7(a) and 2.7(f).

Let us first consider the covariant diagram 2.7(a). From the Feynman rules we obtain

$$\frac{ie^2}{\not{p} - m} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \frac{\not{p}' - \not{k} + m}{(p'-k)^2 - m^2} \gamma^\nu D_{\mu\nu}(k) \frac{1}{\not{p}' - m} . \tag{2.81}$$

We now use (2.69) in the numerator and only retain those terms which have a pole

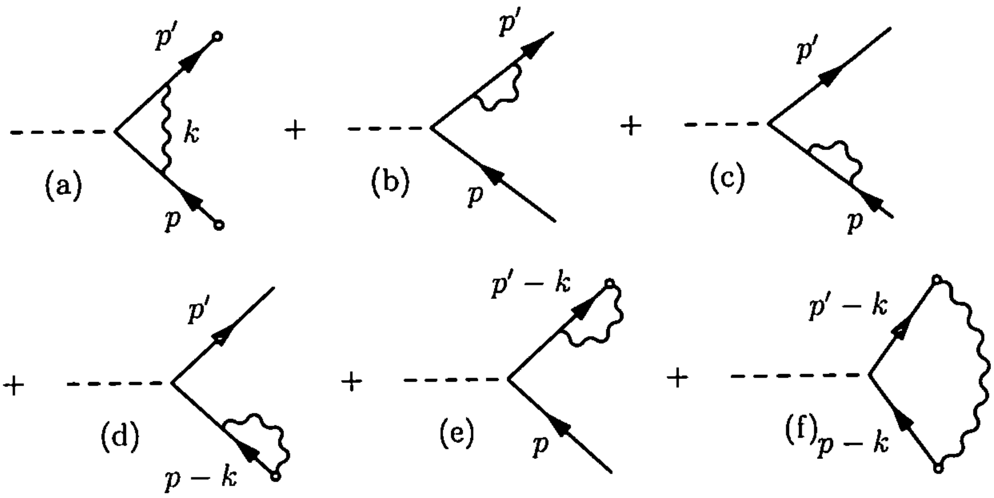


Figure 2.7: *All one-loop Feynman diagrams in the scattering vertex off a current which contain IR-divergences.*

for each external leg and are IR divergent. In this way we obtain

$$\frac{ie^2}{\not{p}-m}\int\frac{d^4k}{(2\pi)^4}\frac{p^\mu}{p\cdot k}\frac{p'^\nu}{p'\cdot k}\frac{1}{\not{p}'-m}D_{\mu\nu}(k)\frac{1}{\not{p}'-m}. \tag{2.82}$$

The contribution of the rainbow diagram 2.7(f) to the vertex, after using the technique shown in [23], is

$$-\frac{ie^2}{\not{p}-m}\int\frac{d^4k}{(2\pi)^4}\frac{V^\mu}{V\cdot k}\frac{V'^\nu}{V'\cdot k}\frac{1}{\not{p}'-m}D_{\mu\nu}(k)\frac{1}{\not{p}'-m}. \tag{2.83}$$

We can now combine all the IR divergent terms associated with the one loop vertex corrections from the diagrams of Figure 2.7 to obtain

$$\begin{aligned}-\frac{ie^2}{\not{p}-m}\int\frac{d^4k}{(2\pi)^4}&\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V'^\mu}{V'\cdot k}\right]D_{\mu\nu}(k)\left[\frac{V^\nu}{V\cdot k}-\frac{p'^\nu}{p'\cdot k}\right]\right.\\&\left.-\left[\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}\right]D_{\mu\nu}(k)\left[\frac{p^\nu}{p\cdot k}-\frac{p'^\nu}{p'\cdot k}\right]\right\}\frac{1}{\not{p}'-m}.\end{aligned} \tag{2.84}$$

This result makes the gauge invariant nature of our dressed Green's function manifest. We can use the argument based on (2.80) to show that the IR divergences cancel when both incoming and outgoing momenta are placed at the correct points on the mass shell. We stress that the vertex diverges in the IR domain if we are not at the correct point on the mass shell. Finally, it has also been shown in [7] that the structure (2.84) is identical in scalar QED once we replace $1/(\not{p}-m)$ by $1/(p^2-m^2)$, which confirms that the IR structures are spin independent.

Having thus demonstrated the cancellation of the various IR divergences which occur in the on-shell residue, we now present a full calculation of the one loop renormalisation constants associated with the physical electron propagator [13, 29].

The Dressed Propagator

We examine the one loop mass shell renormalisation of the electron propagator and for simplicity we use the static dressing. At the static point, the dressing gauge $\mathcal{G}^\mu A_\mu = 0$ with $\mathcal{G}^\mu = (\eta+v)^\mu(\eta-v)\cdot\partial-\partial^\mu$, is essentially the familiar Coulomb gauge. As far as Feynman diagrams are concerned, in the Coulomb gauge only Figure 2.6(b) will contribute. In order to maintain the gauge invariance of our final result we will work in a gauge invariant regularisation scheme, i.e. dimensional regularisation.

To begin the Coulomb gauge propagator is

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left[g_{\mu\nu} - \frac{k\cdot\eta(\eta_\mu k_\nu + k_\mu\eta_\nu)}{k^2} + \frac{k_\mu k_\nu}{k^2}\right]\,. \tag{2.85}$$

The expression for the self-energy in D dimensions in Coulomb gauge, is

$$
\begin{aligned}
-i\Sigma(p) &= e^2 \int \frac{d^Dk}{(2\pi)^D} \left\{ \frac{1}{k^2} \frac{1}{(p-k)^2 - m^2} \left[(D-2)\not{p} - Dm - (D-2)\not{k} \right] \right. \\
&+ \frac{1}{k^2} \frac{1}{(p-k)^2 - m^2} \left[-2(\not{p} - m) \right] \\
&+ \frac{1}{(p-k)^2 - m^2} \frac{1}{k^2 - (k\cdot\eta)^2} \left[(\not{p} - m) + \not{\eta} k\cdot\eta \right] \\
&+ \frac{1}{k^2} \frac{1}{(p-k)^2 - m^2} \frac{1}{k^2 - (k\cdot\eta)^2} \left[-(p^2 - m^2)\not{\eta} k\cdot\eta - 2\not{k} k\cdot\eta p\cdot\eta \right] \\
&+ \left. \frac{1}{k^2} \frac{1}{(p-k)^2 - m^2} \frac{1}{k^2 - (k\cdot\eta)^2} \left[(p^2 - m^2)\not{k} \right] \right\} . \qquad (2.86)
\end{aligned}
$$

In a covariant gauge we must include all the diagrams of Figure 2.6 and it is easy to show that this approach also yields (2.86).

We now proceed to evaluate the above integrals. The calculation of is not completely trivial, due to the non-covariant nature of the integrand. We use the technique shown in [13, 63]. A discussion of this can also be found in Appendix E. As far as mass renormalisation is concerned, we only have three integrals to perform in (2.86), which are the usual covariant ($g^{\mu\nu}$) part, the part with $\not{\eta} k\cdot\eta$ in the numerator and the part with $2\not{k} k\cdot\eta p\cdot\eta$. The remaining integrals have a factor of $(\not{p} - m)$ and as such, they will only contribute to the wave function renormalisation.

To find the mass shift renormalisation constant, δm, we use the mass shell condition (2.13) and, after performing the relevant integrals in (2.86), obtain

$$
\begin{aligned}
\frac{\delta m}{m} &= \frac{e^2}{16\pi^2} \left\{ \left[\frac{3}{\hat{\varepsilon}} + 3\ln\left(\frac{\Lambda^2}{m^2}\right) + 4 \right] \right. \\
&- \left. 2\int_0^1 dx \frac{1}{\sqrt{1-x}} \frac{(p\cdot\eta)^2}{m\Pi} \left[(1-x)m - \Pi \frac{\not{\eta}}{p\cdot\eta} + xp\cdot\eta\not{\eta} \right] \right\} , \qquad (2.87)
\end{aligned}
$$

where $\Pi = (1-x)p^2 + x(p\cdot\eta)^2$. The first term in the above expression is the same as

(2.14), the result for the mass shift in Feynman gauge. In order to show that this is gauge invariant, we now need to see that the other terms all cancel on-shell no matter what exact point of the mass shell is used. To do this we employ the Gordon identity in (2.87). For on-shell spinors, it takes the form

$$\bar{u}(p)\not{\eta}u(p) = \frac{p \cdot \eta}{m}. \tag{2.88}$$

Using this, we find that the mass shift is gauge invariant and we obtain the standard result (2.14).

We now calculate the wave function renormalisation constant (δZ_2) in Coulomb gauge. To do this, we differentiate the self-energy (2.86), with respect to $\not{p}$, then perform the integrals after going on-shell. Since the self-energy is non-covariant we must specify which mass shell point we use. In Coulomb gauge, we evaluate the integral at the static point, $p^\mu = m\eta^\mu$. The full expression for δZ_2 in Coulomb gauge is

$$\delta Z_2 = -\frac{e^2}{16\pi^2}\left\{\left[\frac{1}{\hat{\varepsilon}_{\rm UV}} + \ln\left(\frac{\Lambda^2}{m^2}\right)\right]\right\}. \tag{2.89}$$

Thus we see that there is no IR divergence in the static version of the dressed propagator. This is the same δZ_2 as that found in [63].

2.5 Summary

In this chapter, we have studied the one loop renormalisation of QED and we have discussed the IR divergences that occur in 3+1 dimensions. To cancel the IR divergences we have introduced the Bloch-Nordsieck approach. To understand the physical origin of the IR problem we then introduced the formalism of Kulish and Fadeev [10], which is based on determining the correct form of the asymptotic interaction in gauge theories. They have shown that the idea of switching off the coupling is no longer valid in QED in 3+1 dimensions. KF further showed that this implies that the Lagrangian matter field does not asymptotically approach the free field form of the plane wave expansion. They concluded from this that it is not possible to describe charged particles in QED. However, we would argue that it only shows that any description of a charged particle must be gauge invariant. This is not the case for the Lagrangian fermion in the interacting field theory.

We gave a description of charged particles which involved dressing the matter with the appropriate electromagnetic cloud. We showed that this dressing is composed of two factors: a minimal component which has the correct gauge transformation properties and an additional, gauge invariant part which is necessary to fulfill the dressing equation. In this chapter's explicit calculations we have only used the minimal dressing which is sufficient in 3+1 dimensions to remove IR divergences that usually arise in the on-shell residue of the matter propagator. We have repeated the one loop calculation of the minimally dressed propagator for the static charge. A one loop calculation of the dressed propagator for the moving charge can be found in [13, 29].

To summarise, we saw that there are no IR divergences in the perturbative expansion of the dressed electron propagator at one loop. This is also true at all orders in perturbation theory. A discussion of this can be found in [23]. We conclude that charged particles may indeed be described in QED.

Having now prepared ourselves by studying these various aspects of, and responses to, the IR problem in 3+1 dimensions, we can move on to 2+1 dimensions where naive power counting already tells us that the IR divergences will be harder to handle.

Chapter 3

Quantum Electrodynamics in 2+1 Dimensions

In this chapter we will begin our study of gauge theories in 2+1 dimensions. We will first study the IR behaviour of the on-shell 2-point and 3-point Green's functions in spinor electrodynamics. We shall then perform the analogous calculations in scalar QED to investigate any spin dependence of the IR structures. In both cases we will calculate the renormalisation constants associated with these Green's functions. For the 2-point Green's function (the electron propagator) these are the mass shift and the fermion wave function renormalisation. We shall also study the renormalisation constant associated with the vertex correction, which will enable us to check the

Ward identity. It is well known that the momentum integration associated with these Green's functions can have divergences in both the UV and IR domains. In 2+1 dimensions IR divergences are, as a simple consequence of power counting, worse than in 3+1, and before we implement any renormalisation, we need to know how to regulate such divergences. In this chapter we will consider three different types of IR regularisation schemes: the photon mass scheme, dimensional regularisation and use of a residual off-shellness. To regulate the UV divergences, which as we shall see only occur in the case of scalar electrodynamics, we will use Pauli-Villars regularisation. Since we are dealing with gauge theories, it is important for us to study any gauge dependence. We will do this by calculating the Green's function in different gauges. In particular, we will consider the general Lorentz class of gauges.

3.1 Spinor Electrodynamics

We first consider spinor electrodynamics (QED) and use the Feynman rules described in the previous section to study the matter propagator and the three-point vertex at one loop. We should point out here that the electric charge, e, appearing in the Lagrangian, is a dimensionful quantity in 2+1 dimensions. Once again we will calculate all the relevant renormalisation constants in 2+1 dimensions. (The other primitively divergent diagram in QED is the vacuum polarisation, which is IR finite and is therefore not considered in this thesis.) Since this is a gauge theory, we must ensure that all our results fulfill the Ward identity. We begin by looking at the fermion propagator.

3.1.1 The Matter Propagator

We begin by recalling some facts about the mass-shell renormalisation of the fermion propagator (the matter two-point function) in 2+1 dimensions and setting up our conventions. We then use different regularisation schemes to regulate the divergences. We start in Feynman gauge and then, to study the gauge dependence of the propagator, we will look at other gauges. The perturbative expansion of the matter propagator at one loop is shown diagrammatically in Figure 2.1.

The matter self-energy corresponding to the second diagram in Figure 2.1 is given in Feynman gauge by

$$-i\Sigma(p) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \gamma^\mu \frac{1}{\not{p} - \not{k} - m} \gamma_\mu \,. \tag{3.1}$$

Naive power counting indicates that this diagram can have both UV and on-shell IR divergences. We will see later that it is actually UV finite in 2+1 dimensions. As before, in order for the propagator to be properly renormalised, we require two different types of renormalisation, a mass shift ($m \to m - \delta m$) and the fermion wave function renormalisation ($\psi_B \to \sqrt{Z_2}\psi_R$ with $Z_2 = 1 + \delta Z_2$). The counterterms in the self-energy corresponding to the third diagram in Figure 2.2 are thus given by

$$-i\Sigma^{\text{counter}} = \delta Z_2(\not{p} - m) + i\delta m \,. \tag{3.2}$$

We thus have exactly the same conditions as in 3+1 dimensions:

$$\delta m = \Sigma(p)|_{\not{p}=m} , \tag{3.3}$$

$$\delta Z_2 = \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m} , \tag{3.4}$$

which determine the counterterms.

It is now essential for us to regulate the IR divergences associated with the matter propagator given by (3.1). We shall study three different regularisation schemes:

(a) The photon mass scheme. A small photon mass is introduced to regulate the IR singularities. This is frequently used for IR divergences.

(b) Dimensional regularisation. This is the most common scheme for regularisation but, as we shall see, this scheme sets some of the divergences that occur in 2+1 dimensions to zero and its use in 2+1 dimensions therefore needs to be treated with caution.

(c) The "near mass shell" scheme which regulates the IR divergences by keeping slightly off-shell.

This last scheme is not so widely used but it will be of use to us for comparison with the results obtained from the photon mass scheme. We begin with the photon mass scheme, recalling first some important tools to calculate the 2+1 dimensional integrals which have more than one denominator.

The photon mass scheme

In this regularisation the photon is given a small mass μ, which will then act as an IR cutoff. To do this we redefine the Lagrangian in (2.1) as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{\mu^2}{2}A_\mu^2, \tag{3.5}$$

where we have introduced a photon mass term proportional to μ^2. The IR divergences will reveal themselves as singularities as $\mu \to 0$. With this new Lagrangian we rewrite the photon propagator in Feynman gauge ($\xi = 1$) as follows:

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 - \mu^2}\,. \tag{3.6}$$

To carry out the integrals, we use the Feynman trick to rewrite the denominators in (3.1) as follows:

$$\begin{aligned} \frac{1}{(k^2 - \mu^2)[(p-k)^2 - m^2]} &= \int_0^1 dx \frac{1}{[(1-x)k^2 + x((p-k)^2 - m^2)]^2} \\ &= \int_0^1 dx \frac{1}{[(k - xp)^2 - m^2 x + p^2 x(1-x) - \mu^2(1-x)]^2}\,. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.1) and shifting $k \to k + xp$, we obtain

$$-i\Sigma(p) = -e^2 \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \frac{-(1-x)\not{p} + 3m}{(k^2 - a)^2}\,, \tag{3.8}$$

where we have dropped the odd integral in k and defined a as

$$a = m^2 x - p^2(1-x) + \mu^2(1-x)\,. \tag{3.9}$$

We now see that the denominator depends only on k^2 and integrating over d^3k will be much easier, since the integrand is spherically symmetric with respect to k. To carry out the momentum integral we perform a Wick rotation from three dimensional Minkowski space to three dimensional Euclidean space, i.e.,

$$k^0 \to ik^0 \quad \text{and} \quad k^i = k^i\,. \tag{3.10}$$

Our rotated contour goes from $k^0 = -\infty$ to ∞. By changing variables to Euclidean 3-momentum, k, we can now evaluate the integral in three-dimensional spherical coordinates. We will need the formula for the volume in D Euclidean dimensions, i.e.,

$$\int f(k^2)d^Dk = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty f(k^2)k^{D-1}dk\,. \tag{3.11}$$

Applying (3.10) and (3.11) to (3.8), we obtain

$$-i\Sigma(\not{p}) = \frac{ie^2}{8\pi} \int_0^1 dx \frac{(1-x)\not{p} - 3m}{(a)^{\frac{1}{2}}}\,. \tag{3.12}$$

After going on-shell and performing the x integration, we obtain the following expression for the electron self-energy,

$$-i\Sigma(\not{p} = m) = -\frac{ie^2}{4\pi}\left[\ln\left(\frac{\mu}{m}\right) - \ln(2) - \frac{1}{2}\right]. \tag{3.13}$$

To calculate δm, we use (3.3) and obtain

$$\delta m = -\frac{e^2}{4\pi}\left[\ln\left(\frac{\mu}{m}\right) - \ln(2) - \frac{1}{2}\right]. \tag{3.14}$$

In the last two equations we have expanded around $\mu \approx 0$ to extract the IR divergent and finite terms. Note that the mass shift, δm, is logarithmically IR divergent in 2+1 dimensions, but IR finite in 3+1 dimensions. To calculate the wave function renormalisation of the matter field, δZ_2, we differentiate (3.12) with respect to $\not{p}$ and then use (3.3). We obtain

$$\delta Z_2 = \frac{e^2}{4\pi}\frac{1}{m}\left[\frac{m}{\mu} + \ln\left(\frac{\mu}{m}\right) - \ln(2) - \frac{1}{2}\right] . \tag{3.15}$$

This is the complete expression for δZ_2, which includes both finite and infinite corrections. Note that in δZ_2 we now have linear IR infinities as well as the logarithmic singularity familiar from 3+1 dimensional QED. The IR divergence parts of these results agree with Sen's work in [64], though Sen did not calculate the finite parts. We now repeat the above calculation using dimensional regularisation.

Dimensional Regularisation

We have shown in Appendix D that dimensional regularisation regulates logarithmic divergences (as $1/\epsilon$ poles) and sets all power divergences to zero. Similar arguments have also been shown in [65] and [66] (see also Section 4.2 in [67]). This is because dimensional regularisation does not introduce a scale which would be needed to reproduce power divergences. This leads to the question of whether or not this is a problem or an advantage of the method. Such a question is especially interesting for us in three dimensions where on-shell Green's functions appear to have linear IR divergences (e.g., the reciprocal of a (small) photon mass, or a residual off-shellness).

Dimensional regularisation simply sets these divergences to zero. This is sometimes claimed to be an advantage of the scheme [65, 66].

We note, however, that it has been argued [68–70] that UV quadratic divergences can indeed be analysed in dimensional regularisation. In particular it has been used to study quadratic UV divergences in four dimensional theories. This involved the use of counterterms designed to make the theory additionally finite in dimensions $D < 4$. It would be interesting to try to extend such an approach to the linear IR singularities which our direct application of dimensional regularisation will set to zero.

Let us consider the fermion self-energy in Feynman gauge. In D dimensions this takes the form

$$-i\Sigma(p) = -e^2(\Lambda)^{3-D} \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{(2-D)(1-x)\not{p} + Dm}{(k^2-a)^2}, \tag{3.16}$$

where we have introduced the Feynman parameter and have dropped all the odd integrals after shifting the k integration. Here a in the denominator is given by (3.9) with $\mu = 0$. The factor $(\Lambda)^{3-D}$ is needed since, in D dimensions, we must multiply e by $\Lambda^{(3-D)/2}$, where Λ is an arbitrary mass scale. The dimension D is defined as

$$D = 3 - 2\varepsilon\,. \tag{3.17}$$

We now use the standard formula (E.1) from Appendix E for integrating over arbitrary dimensions to obtain

$$-i\Sigma(\not p) = \frac{ie^2}{16\pi^2}(\Lambda)^{3-D}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 dx\,[(D-2)(1-x)\not p - Dm]\left(\frac{4\pi m^2}{m^2x - p^2x(1-x)}\right)^{2-\frac{D}{2}} . \tag{3.18}$$

To calculate δm we go on-shell and then use (3.3). We obtain

$$\delta m = -\frac{e^2}{16\pi^2}(\Lambda)^{3-D}\left(\frac{4\pi}{m^2}\right)^{(2-\frac{D}{2})}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 dx\,\frac{m[(D-2)(1-x)-D]}{x^{4-D}} . \tag{3.19}$$

Going on-shell and carrying out the k-integral leads to a divergent Feynman parameter integral in the mass shift of the form:

$$\int_0^1 dx \frac{1+\mathcal{O}(x)}{x^{4-D}} , \tag{3.20}$$

which is regulated by using (3.17) and leads to poles in $1/\epsilon$. After performing the x integration the expression for δm becomes

$$\delta m = -\frac{e^2}{4\pi}\left(\frac{1}{2\hat{\varepsilon}} - \ln(2) + \ln\left(\frac{\Lambda}{m}\right) - \frac{1}{2}\right) . \tag{3.21}$$

To calculate δZ_2 we now differentiate (3.18) with respect to $\not p$ and then apply (3.3), yielding

$$\delta Z_2 = -\frac{e^2}{16\pi^2}(\Lambda)^{3-D}\left(\frac{4\pi}{m^2}\right)^{(2-\frac{D}{2})}\Gamma\left(2-\frac{D}{2}\right)\int_0^1 dx\left\{\left(\frac{[(D-2)(1-x)]}{x^{4-D}}\right) + \left(\frac{(4-D)x(1-x)[(D-2)(1-x)-D]}{x^{5-D}}\right)\right\} . \tag{3.22}$$

In δZ_2, we now have Feynman parameter integrals of the form

$$\int_0^1 dx \frac{1+\mathcal{O}(x)}{x^{5-D}} . \tag{3.23}$$

The terms of order x are now IR logarithms while higher powers of x are finite in $D = 3$. Initially we neglect these and look at the most divergent term (the term with no x's in the numerator).

If we were to carry out the integration for $D = 3 - 2\varepsilon$ and assume ε to be small the integral would diverge around $x \approx 0$. However, in the spirit of dimensional regularisation, we carry out the integral for D so that the integral is finite and then take $D \to 3$. For the terms that would appear to be linearly divergent, this yields

$$\int_0^1 dx \frac{1}{x^{5-D}} = \frac{1}{D-4} \to -1\,. \tag{3.24}$$

In other words dimensional regularisation implies that the integral is finite. Using this argument and then performing the remaining x integration, we obtain after expanding over small ε,

$$\delta Z_2 = \frac{\mathrm{e}^2}{4\pi}\frac{1}{m}\left(\frac{1}{2\hat{\varepsilon}} - \ln(2) + \ln\left(\frac{\Lambda}{m}\right) - \frac{1}{2}\right)\,. \tag{3.25}$$

The IR divergent term that appears in δZ_2 corresponds to the logarithmic IR infinities in (3.15). Once again we have to ask ourselves whether this is a positive feature of the scheme or something we ought to worry about.

We finish this subsection by calculating the renormalisation constants associated with the electron propagator using the near mass shell scheme.

Near mass shell scheme

An alternative way to regulate the IR divergence is to stay "slightly" off-shell, i.e., regulate the divergent integrals at $p^2 = m^2 - \Delta^2$ and then take the limit $\Delta \to 0$, to extract the IR divergences. To calculate δm and δZ_2 we proceed along the lines of the photon mass scheme; we use the Feynman trick to combine the denominators, perform the momentum integration and then, after going on-shell, perform the parametric integration. After some algebra, we obtain

$$\delta m = -\frac{e^2}{4\pi}\left[\ln\left(\frac{\Delta}{m}\right) - \ln(2) - \frac{1}{2}\right], \tag{3.26}$$

and

$$\delta Z_2 = \frac{e^2}{4\pi}\frac{1}{m}\left[\frac{m}{\Delta} + \ln\left(\frac{\Delta}{m}\right) - \ln(2) + 1\right]. \tag{3.27}$$

Comparing with the results for the small photon mass scheme, we see that the expressions for the mass shift and the IR divergent part of the wave function renormalisation are identical. This completes our calculations for the renormalisation constants associated with the electron propagator in Feynman gauge. For ease of comparison we list the results for these constants in the different regularisation schemes.

	δm	δZ_2
Photon mass	$-\frac{e^2}{4\pi}\left[\ln\left(\frac{\mu}{m}\right) - \ln(2) - \frac{1}{2}\right]$	$\frac{e^2}{4\pi}\frac{1}{m}\left[\frac{m}{\mu} + \ln\left(\frac{\mu}{m}\right) - \ln(2) - \frac{1}{2}\right]$
Near mass shell	$-\frac{e^2}{4\pi}\left[\ln\left(\frac{\Delta}{m}\right) - \ln(2) - \frac{1}{2}\right]$	$\frac{e^2}{4\pi}\frac{1}{m}\left[\frac{m}{\Delta} + \ln\left(\frac{\Delta}{m}\right) - \ln(2) + 1\right]$
Dim Reg	$-\frac{e^2}{4\pi}\left(\frac{1}{2\hat{\varepsilon}} - \ln(2) - \frac{1}{2}\right)$	$\frac{e^2}{4\pi}\left(\frac{1}{2\hat{\varepsilon}} - \ln(2) - \frac{1}{2}\right)$

From this table we see that the renormalisation constants δm and δZ_2 can be transformed from one regularisation scheme into another via a dictionary but with the important proviso that dimensional regularisation sets the linear divergence to zero.

Lorentz Gauge

Since this is a gauge theory we should examine the gauge invariance of these parameters. One way to do this is by calculating these quantities in an arbitrary Lorentz gauge. There is a simple, formal argument (sketched, for example, by D. Sen [64] before his Eq. 6) that the mass shift is gauge invariant. It runs as follows: consider the photon propagator

$$D_{\mu\nu}(k) = -\frac{i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right), \tag{3.28}$$

from this the gauge parameter dependent part of the fermionic matter self energy at one loop is essentially

$$-i\Sigma_\xi(\not{p}) = -(1-\xi)e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{[(p+k)^2-m^2]}\frac{k_\mu k_\nu}{k^4}. \tag{3.29}$$

However, we can rewrite

$$\not{k}(\not{p}+\not{k}+m)\not{k} = (2p\cdot k + k^2)\not{k} - (\not{p}-m)k^2, \tag{3.30}$$

which can be substituted back into the *on-shell* self energy (3.29) to yield

$$-i\Sigma(\not{p}) = -(1-\xi)e^2 \int \frac{d^3k}{(2\pi)^3}\left\{\frac{\not{k}}{k^4} - (\not{p}-m)\frac{\not{k}}{[2p\cdot k + k^2]k^2}\right\}. \tag{3.31}$$

Thus it is argued, according to Sen, that the first term is *odd* and must vanish, while the second term does not contribute to the mass shift due to the factor of $(\not{p} - m)$. This implies that the mass shift is gauge invariant.

This argument does not work if one calculates the integral corresponding to (3.29). In the class of covariant gauges, the Feynman gauge IR logarithmic divergence is modified by a gauge dependent finite amount. This is due to IR singularities in the integral was first calculated by Deser, Jackiw and Templeton in [40] (see their Eq. 2.68b), where these authors argued that the correct response was to work in Landau gauge, when the longitudinal component of the vector fields is set to zero.

Sen [64] proceeds with the usual theory by introducing a small photon mass and, since the various integrals in the mass shift are now finite, reasons that the above argument leading to (3.31) holds. He claims that the mass shift is gauge invariant and then works in Feynman gauge to simplify the calculation. As we have seen earlier, the IR divergent mass shift is of the form $e^2 \ln(\mu)/4\pi$.

More recently Hoshino (in [71]) has followed an old work of Jackiw and Soloviev [72] and taken a spectral function approach to the scalar and spinor propagators in 2+1 dimensions. He obtains a gauge dependent mass shift (which is Eq. 44 in [71]).

The immediate question then is, *what is the best approach to the mass shift?* If one stays slightly off-shell, i.e., uses the near mass shell as an IR regulator, we obtain the following results:

$$\delta m = -\frac{e^2}{4\pi}\left[\ln\left(\frac{\Delta}{m}\right) - \ln(2) - \frac{\xi}{2}\right], \tag{3.32}$$

and

$$\delta Z_2 = \frac{e^2}{4\pi}\frac{1}{m}\left[\ln\left(\frac{\Delta}{m}\right) + \frac{m}{\Delta} - \ln(4) + \xi\right], \tag{3.33}$$

which are clearly gauge parameter dependent. As $\Delta \to 0$, they also have the IR divergence and Deser et al's *gauge dependent* finite part. However, at $\xi = 1$ they are equal to (3.26) and (3.27) respectively, which confirms the consistency of our calculations.

We now use a small photon mass as a regulator. There are two ways to perform the calculation. The first is in the spirit of the above argument: go on-shell and then perform the integration. This yields a gauge invariant (logarithmically IR divergent) mass shift. The second is to perform the integrals, then expand in small μ and finally go on-shell. This yields the result of Jackiw et al.

To try to resolve this puzzle we will consider a general non-covariant gauge to study the gauge invariance of the mass shift in fermionic theory .

General Gauge

We first show that if we use the formal argument as described, for example, in [40] and [64] then the mass shift is gauge invariant . The photon propagator in a general non-covariant gauge with gauge fixing $\frac{1}{2\xi}(N \cdot A)^2$ is, in the limit $\xi \to 0$,

$$D_{\mu\nu} = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu N_\nu + N_\mu k_\nu}{k \cdot N} + \frac{N^2 k_\mu k_\nu}{(k \cdot N)^2}\right) . \tag{3.34}$$

We now have two different gauge dependent parts of the electron self energy. The last part, which is proportional to $k_\mu k_\nu$, is essentially the same as the Lorentz class and, therefore, is gauge invariant. We only need to check the part of the term which is proportional to $k_\mu N_\nu + k_\nu N_\mu$. The contribution of this to the self energy is

$$-e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu}{[(p+k)^2 - m^2]k^2} \frac{(k_\mu N_\nu + N_\mu k_\nu)}{k \cdot N} . \tag{3.35}$$

After some algebra, the term in the numerator can be written as

$$\not{k}(\not{p}+\not{k}+m)\not{N}+\not{N}(\not{p}+\not{k}+m)\not{k} = 2N\cdot k(-\not{p}+m)+2p\cdot k\not{N}+2k^2\not{N}+2\not{k}N\cdot p . \tag{3.36}$$

Substituting this into (3.35) and then going on-shell yields,

$$-e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{[k^2 + 2p \cdot k]k^2 k \cdot N} \left[-(\not{p} - m)2N \cdot k + 2p \cdot k\not{N} + 2k^2\not{N} + 2\not{k}N \cdot p\right] . \tag{3.37}$$

The first term in the above expression does not contribute to the mass shift due to the factor $(\not{p} - m)$. If we use the Gordon identity in the remaining terms, we obtain

$$\begin{aligned} -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{[k^2 + 2p \cdot k]k^2 k \cdot N} &[2p \cdot k\not{N} + 2k^2\not{N} + 2\not{k}N \cdot p] \\ &= -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 k \cdot N} \frac{2N \cdot p}{m} , \end{aligned} \tag{3.38}$$

which is clearly an *odd* integral and therefore can be dropped. This seems to imply that the mass shift is gauge invariant. However, no regularisation has yet been introduced.

We now use a small photon mass as a regulator and see what overall result is obtained. For a general non-covariant gauge we similarly obtain the propagator in the limit $\xi \to 0$:

$$D_{\mu\nu} = \frac{1}{k^2-\mu^2}\left(g_{\mu\nu} - \frac{(k_\mu N_\nu + N_\mu k_\nu)k\cdot N}{(k\cdot N)^2 - \mu^2 N^2} + \frac{N^2 k_\mu k_\nu}{(k\cdot N)^2-\mu^2N^2} + \frac{\mu^2 N_\mu N_\nu}{(k\cdot N)^2 - \mu^2 N^2}\right), \tag{3.39}$$

which may be easily checked to obey $N^\mu D_{\mu\nu} = 0$. The additional tensor term $N_\mu N_\nu$ in the propagator, is a consequence of using a gauge fixed Lagrangian with a small mass term.

We now choose $N^\mu = m\gamma(\eta + v)^\mu$, which is essentially equivalent to a dressing gauge $N \cdot A = 0$, as discussed in the previous chapter. For simplicity we will specialise to the static point on the mass shell, $N = m(1,0,0)$, though this can easily be generalised. Following Sen's argument the middle tensor structures in (3.39) will not contribute to the mass shift. The Feynman gauge structure ($g_{\mu\nu}$) leads to the result

$$-i\Sigma^{\mathrm{F}} = -e^2\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\left\{\frac{2m}{2p\cdot k(k^2-\mu^2)}\right\}, \tag{3.40}$$

plus sub-leading terms and structures with only one pole. The additional $N_\mu N_\nu$ structure in (3.39) similarly generates an additional contribution

$$-i\Sigma^{\mathrm{extra}} = -e^2\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\left\{\mu^2\frac{2m}{2p\cdot k(k^2-\mu^2)(mk_0-\mu^2)}\right\}, \tag{3.41}$$

plus sub-leading contributions. Although the integral here is multiplied by μ^2, it is evidently more divergent in the IR region than the integral in (3.40).

We have calculated both of these terms as follows. In (3.40), with

$$k^2 - \mu^2 + i\epsilon = (k_0 - \sqrt{\boldsymbol{k}^2 + \mu^2} + i\epsilon)(k_0 + \sqrt{\boldsymbol{k}^2 + \mu^2} - i\epsilon) \tag{3.42}$$

and integrating k_0 over the upper half plane to avoid the pole in $p \cdot k + i\epsilon$, we obtain, from Cauchy's theorem,

$$-i\Sigma^{\mathrm{F}} = -ie^2 \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \frac{1}{\boldsymbol{k}^2 + \mu^2}\,, \tag{3.43}$$

which yields the IR logarithmic divergence

$$-i\Sigma^{\mathrm{F}} = +\frac{ie^2}{4\pi} \ln(\mu)\,. \tag{3.44}$$

This is the standard Feynman gauge result.

The new correction can be calculated in the same way although there are now more poles to be taken into account. Picking up two contributions, we find

$$-i\Sigma^{\mathrm{extra}} = -\frac{ie^2\mu^2}{4m} \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \left[\frac{2m - \sqrt{\boldsymbol{k}^2 + \mu^2}}{(\boldsymbol{k}^2 + \mu^2)\boldsymbol{k}^2} - \frac{2m - \mu}{\boldsymbol{k}^2\mu^2}\right]. \tag{3.45}$$

Combining these terms and carrying out the $\boldsymbol{k}$ integral leads to a logarithmic divergence but by adding $\Sigma^{\mathrm{F}} + \Sigma^{\mathrm{extra}}$ we obtain an *IR finite result.*

In summary, it seems that although using a small photon mass at the level of the Lagrangian to regulate the IR divergences leaves the mass shift gauge invariant in the class of covariant gauges, the mass does pick up a gauge dependence in non-covariant gauges as a result of the new $N_\mu N_\nu$ tensor structure. In particular, we conclude that the propagator in the above gauge, with a small photon mass introduced as an IR regulator, is IR finite as long as the on-shell momentum and this class of dressing gauge fixing parameter are equivalent.

Having thus calculated the various expressions for the renormalisation constants associated with the electron propagator, we move on to the vertex correction diagram.

3.1.2 The Vertex Correction

Consider the three-point vertex correction, given by the following diagram

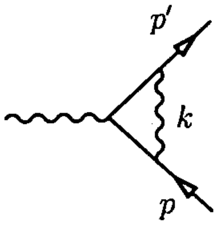


Figure 3.1: *The vertex correction diagram.*

This case is harder to calculate than the propagator since three internal propagators are involved. Employing the Feynman rules described in the previous section, we can write the one loop vertex correction in arbitrary D dimensions and in Feynman gauge, (which we shall use throughout our vertex calculations) as follows:

$$\Gamma^{\mu}_{(1)}(p,p') = (ie)^2 \int \frac{d^D k}{(2\pi)^D} \gamma^{\rho} \frac{i}{\not{p}' - \not{k} - m} \gamma^{\mu} \frac{i}{\not{p} - \not{k} - m} \gamma_{\rho} \frac{-i}{k^2}, \tag{3.46}$$

$$= -ie^2 \int \frac{d^D k}{(2\pi)^3} \frac{\gamma^{\rho}(\not{p}' - \not{k} + m)\gamma^{\mu}(\not{p} - \not{k} + m)\gamma_{\rho}}{((p-k)^2 - m^2)((p'-k)^2 - m^2)k^2}. \tag{3.47}$$

For on-shell external momenta, we may write

$$\gamma^\rho(\not{p}' - \not{k} + m) = 2p'^\rho - \gamma^\rho \not{k}\,, \tag{3.48}$$

and

$$(\not{p} - \not{k} + m)\gamma_\rho = 2p^\rho - \not{k}\gamma^\rho\,. \tag{3.49}$$

Substituting these into (3.46) we obtain, after a little bit of algebra,

$$\Gamma^\mu_{(1)}(p,p') = -ie^2[4p\cdot p'\gamma^\mu I_0 - 2(\not{p}\gamma_\alpha\gamma^\mu + \gamma^\mu\gamma_\alpha\not{p}')I_1^\alpha + (D-2)(g_{\alpha\beta}\gamma^\mu - 2\gamma_\alpha g^\mu_\beta)I_2^{\alpha\beta}]\,, \tag{3.50}$$

where

$$\begin{aligned} I_0 &= \int \frac{d^Dk}{(2\pi)^D}\frac{1}{((p-k)^2-m^2)((p'-k)^2-m^2)k^2}\,, \\ I_1^\alpha &= \int \frac{d^Dk}{(2\pi)^D}\frac{k^\alpha}{((p-k)^2-m^2)((p'-k)^2-m^2)k^2} \equiv A(p+p')^\alpha\,, \\ I_2^{\alpha\beta} &= \int \frac{d^Dk}{(2\pi)^D}\frac{k^\alpha k^\beta}{((p-k)^2-m^2)((p'-k)^2-m^2)k^2} \\ &\equiv Bg^{\alpha\beta} + C(p^\alpha p^\beta + p'^\alpha p'^\beta) + E(p^\alpha p'^\beta + p'^\alpha p^\beta)\,. \end{aligned} \tag{3.51}$$

The coefficients A, B, C, and E can be found by evaluating the above integrals. Using these definitions for I_0, I_1^α and $I_2^{\alpha\beta}$, we find that the vertex correction is,

$$\Gamma^\mu_{(1)}(p,p') = \alpha\gamma^\mu + \beta(p+p')^\mu\,, \tag{3.52}$$

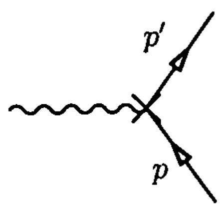
where

$$\begin{aligned} \alpha(p-p') &= -ie^2(4p\cdot p'I_0 - 4Am^2 + (D-2)^2B + 2(D-2)Cm^2 \\ &\quad +2(D-2)p\cdot p'E - A(4m^2+8p\cdot p')) \end{aligned} \tag{3.53}$$

and

$$\beta(p-p') = -ie^2(4Am - 2(D-2)m(C+E)) \,. \tag{3.54}$$

Our aim is to calculate δZ_1, the renormalisation parameter associated with the vertex. The Feynman rule for the vertex counter term is shown below in Figure 3.2.



$$= \; ie\gamma^\mu \delta Z_1 \,.$$

Figure 3.2: *The counter term diagram for the vertex correction.*

The renormalisation condition associated with the vertex correction is

$$\Gamma^\mu_{(1)}(p-p'=0) = \gamma^\mu \,. \tag{3.55}$$

Now apply the Gordon identity in (3.52), which is valid between on shell spinors,

$$\gamma^\mu - \frac{i}{2m}\sigma^{\mu\nu}(p+p')_\nu = \frac{1}{2m}(p+p')^\mu \,. \tag{3.56}$$

We thus obtain

$$\Gamma^\mu_{(1)}(p,p') = (\alpha + 2m\beta)\gamma^\mu + \frac{i}{2m}\sigma^{\mu\nu}(p+p')_\nu(-2m\beta) \,. \tag{3.57}$$

Using (3.55) and (3.57), we obtain the on-shell vertex renormalisation condition:

$$\delta Z_1 = -(\alpha(p-p') + 2m\beta(p-p')) \,, \tag{3.58}$$

where

$$\alpha(p-p') + 2m\beta(p-p') = -ie^2(4p\cdot p' I_0 - 8Ap\cdot p' + (D-2)^2 B \\ -2(D-2)Cm^2 + 2(D-2)E(p\cdot p' - 2m^2)) . \quad (3.59)$$

At zero momentum transfer, $p \cdot p' = m^2$ and hence

$$\alpha + 2m\beta = -ie^2(4m^2 I_0 + (D-2)^2 B - 2(D-2)m^2(C+E) - 8Am^2) . \quad (3.60)$$

In order to calculate I_0, A, B, C and E, we need to evaluate the integrals (3.51), (3.52) and (3.53). We will see that a divergence arises in A only. All the others make only finite contributions to the vertex. As with the electron propagator we will use two different regulators beginning with dimensional regularisation.

Dimensional Regularisation

Using the techniques of the previous section and the formulae of Appendix E, we proceed to evaluate the integrals in (3.51). As before, we multiply e by $\Lambda^{(3-D)/2}$, where Λ is an arbitrary mass and define $D = 3 - 2\varepsilon$. Notice that in the vertex we have three terms in the denominator and the Feynman trick for three denominators is

$$\frac{1}{\mathcal{ABC}} = 2\int_0^1 dy \int_0^1 dx \, \frac{x}{[(\mathcal{A}-\mathcal{B})xy + (\mathcal{B}-\mathcal{C})x + \mathcal{C}]^3} . \quad (3.61)$$

Here, $\mathcal{A} = (p-k)^2 - m^2$, $\mathcal{B} = (p'-k)^2 - m^2$ and $\mathcal{C} = k^2$. For on-shell external momenta, we write $\mathcal{A} - \mathcal{B} = 2(p-p')\cdot k$ and $\mathcal{B} - \mathcal{C} = -2p\cdot k$, so for zero momentum transfer, we have $\mathcal{A} - \mathcal{B} = 0$ and $\mathcal{B} - \mathcal{C} = -2p\cdot k$.

Consider I_0 and introduce two Feynman parameters,

$$I_0 = 2\int_0^1 dy \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{x}{(k^2 - 2p\cdot kx)^3} . \tag{3.62}$$

Changing to the new integration variable $k' = k + px$ (3.62) becomes

$$I_0 = 2\int_0^1 dy \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D} \frac{x}{(k'^2 - p^2x^2)^3} . \tag{3.63}$$

Integrating this over arbitrary dimensions using the formula (E1) from Appendix E and then performing the Feynman parameter integrations, we obtain

$$I_0 = \frac{i}{16\pi^2}(4\pi)^{\frac{1}{2}+\varepsilon}\Gamma\left(\frac{3}{2}+\varepsilon\right)\frac{1}{1+2\varepsilon} . \tag{3.64}$$

Everything in (3.64) is finite, so it is safe to take the limit $\varepsilon \to 0$, giving

$$I_0 = \frac{i}{16\pi}\frac{1}{m^3} . \tag{3.65}$$

The next integral is I_1^α, i.e.,

$$I_1^\alpha = \int \frac{d^D k}{(2\pi)^D} \frac{k^\alpha}{((p-k)^2 - m^2)((p'-k)^2 - m^2)k^2} \equiv A(p+p')^\alpha . \tag{3.66}$$

In order to evaluate this integral, we first introduce two Feynman parameters and, after shifting the k variable, we obtain,

$$I_1^\alpha = 2\int_0^1 dy \int_0^1 dx\, x \int \frac{d^D k}{(2\pi)^D} \frac{xp^\alpha}{(k^2 - x^2m^2)^3} , \tag{3.67}$$

where we have dropped the odd integrals. Here we are on-shell and have taken the limit of zero momentum transfer. We now perform all the necessary integrals, expand over small ε and find

$$I_1^\alpha = \frac{i}{16\pi}\frac{p^\alpha}{2m^3}\left[\frac{1}{\hat{\varepsilon}} - \ln(4) + 2\right] , \tag{3.68}$$

Small Photon Mass

In order to regulate the vertex with a small photon mass, we replace the photon propagator $1/k^2$ with $1/(k^2 - \mu^2)$ and calculate the integrals in 2+1 dimensions. In this setting, we can rewrite the integrals (3.51) as

$$\begin{aligned} I_0 &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{((p-k)^2 - m^2)^2 (k^2 - \mu^2)}, \\ I_1^\alpha &= \int \frac{d^3k}{(2\pi)^3} \frac{k^\alpha}{((p-k)^2 - m^2)^2 (k^2 - \mu^2)} \equiv 2Ap^\alpha, \\ I_2^{\alpha\beta} &= \int \frac{d^3k}{(2\pi)^3} \frac{k^\alpha k^\beta}{((p-k)^2 - m^2)^2 (k^2 - \mu^2)} \equiv Bg^{\alpha\beta} + 2(C+E)p^\alpha p^\beta. \end{aligned} \tag{3.74}$$

Here we have taken the limit $p = p'$, since we wish to calculate δZ_1 and this is evaluated in the limit of zero momentum transfer. We follow the previous procedure to calculate the coefficients I_0, A, B, C and E. After these steps we obtain

$$\begin{aligned} I_0 &= -\frac{i}{16\pi}\frac{1}{m^3}\left[\frac{m}{\mu} - 1\right], \\ A &= \frac{i}{16\pi}\frac{1}{2m^3}\left[\ln\left(\frac{m}{\mu}\right) + 1 - \ln(2)\right], \\ B &= \frac{i}{16\pi}\frac{1}{m} \quad \text{and} \quad C + E = -\frac{i}{16\pi}\frac{1}{2m^3}. \end{aligned} \tag{3.75}$$

Substituting these in (3.58), we obtain

$$\delta Z_1 = \frac{e^2}{4\pi}\frac{1}{m}\left[\ln\left(\frac{\mu}{m}\right) + \frac{m}{\mu} - \ln(2) - \frac{1}{2}\right]. \tag{3.76}$$

This is the same as (3.15), the expression for δZ_2 in the small photon mass scheme, and hence this scheme also preserves the Ward identity.

This finishes our calculations of the renormalisation parameters associated with the 2-point and 3-point Green's functions in spinor electrodynamics in 2+1 dimensions. In the next sections we will repeat the above calculations in scalar theory.

3.2 Scalar Electrodynamics

Scalar electrodynamics, being a theory of spin zero charged particles, does not require the use of gamma matrices, and is therefore algebraically easier to handle. It does though introduce a four point vertex in the Lagrangian. We can compare the results of this theory with those of fermionic quantum electrodynamics, and hence extract any spin dependence. Recall from Chapter 2 that in 3+1 dimensions the IR structures are spin independent. We start from the Lagrangian of a free charged scalar field, ϕ, which is given by (see also Section 6.1.4 of [73])

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \,. \tag{3.77}$$

We replace $\partial_\mu \phi$ by the covariant derivative $(\partial_\mu + ieA_\mu)\phi$ and add the electromagnetic Lagrangian to obtain

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (\partial_\mu \phi + ieA_\mu \phi)(\partial_\mu \phi^* + ieA_\mu \phi^*) - m^2 \phi^* \phi + \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \,. \tag{3.78}$$

The last term in (3.78) is the gauge fixing term. The Feynman rules for this model are shown in Figure 3.3. We follow the conventions used in Itzykson and Zuber [73] except for the sign of the electric charge, e, which we choose to be positive. The photon propagator is

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right], \tag{3.79}$$

and the matter propagator is

$$S(p) = \frac{1}{p^2 - m^2}\,. \tag{3.80}$$

For the three point vertex we write

$$\Gamma^\mu \;=\; \mathrm{i}e(p+p')^\mu\,. \tag{3.81}$$

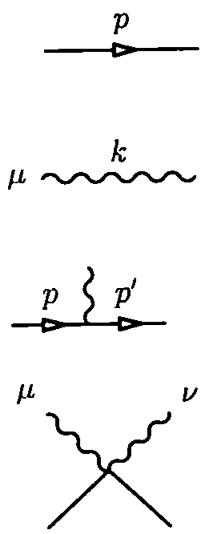
Finally, for the four point vertex we write

$$\Gamma^{\mu\nu} \;=\; 2\mathrm{i}e^2 g^{\mu\nu}\,. \tag{3.82}$$

As in the case of the fermionic theory, we will study the matter propagator and three-point vertex at one loop and calculate all the renormalisation constants associated with them in 2+1 dimensions. Since this is a gauge theory, we must also study the gauge symmetry. We begin by looking at the matter propagator.

3.2.1 The Matter Propagator

At one loop we consider the diagram in Figure 3.4. Unlike the fermionic theory, there is a massless tadpole diagram in scalar QED, which is shown in Figure 3.4b. From



$$= iS(p).$$

$$= -iD_{\mu\nu}$$

$$= ie(p+p')_\mu$$

$$= 2ie^2 g^{\mu\nu}.$$

Figure 3.3: *The Feynman rules for scalar QED.*

the Feynman rules we obtain

$$-i\Sigma(p) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{g_{\mu\nu}}{k^2} \left[-\frac{(2p-k)^\mu (2p-k)^\nu}{(p-k)^2 - m^2} + g^{\mu\nu} \right]. \tag{3.83}$$

Simple power counting ensures that the first integral in the square bracket has both a UV divergence for large k, arising from the integral,

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu k^\nu}{k^2(k^2 - 2p\cdot k)} \tag{3.84}$$

and an IR divergence, on-shell, for small k. The second integral has only a UV divergence.

Massless tadpoles like diagram 3.4(b) are zero if we choose dimensional regularisation to regularise the integrals (see for example [7] and [23]), but they do not generally vanish in other schemes.

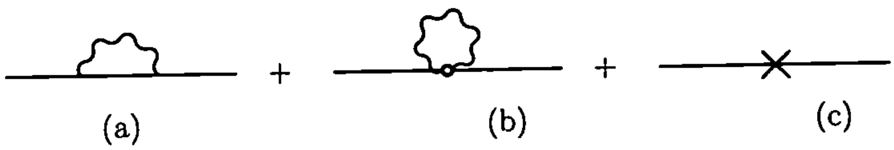


Figure 3.4: *Lowest order diagrams for the scalar matter propagator.*

To renormalise the electron propagator, we still require two different types of renormalisation, a mass shift (δm) and the wave function renormalisation (δZ_2). In scalar QED, we define the counterterms in the self-energy corresponding to the diagram in Figure 3.4(c) as follows:

$$-i\Sigma^{\text{counter}} = \delta Z_2(p^2 - m^2) + i\,2m\delta m\,. \tag{3.85}$$

For on-shell renormalisation, we use the following equations to calculate δm and δZ_2:

$$2m\,\delta m = \Sigma^R(p)\Big|_{p^2=m^2}\,, \tag{3.86}$$

$$\delta Z_2 = \frac{d\Sigma^R}{dp^2}\Big|_{p^2=m^2}\,, \tag{3.87}$$

which should be compared with (3.3) and (3.4) in spinor QED. We wish to regulate divergences using three different regularisation schemes. In scalar QED we now have UV divergences and to regulate these we will employ the Pauli-Villars method as described in Appendix B. To be consistent, let us first consider the IR divergences and follow the steps taken in the fermionic theory.

The photon mass scheme

We start by redefining the Lagrangian in (3.78) as

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu^2}{2}A_\mu^2 + (\partial_\mu\phi + ieA_\mu\phi)(\partial_\mu\phi^* + ieA_\mu\phi^*) - m^2\phi^*\phi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 , \quad (3.88)$$

where we have introduced a photon mass term proportional to μ^2. In order to extract the IR divergences we will take the limit $\mu \to 0$ at the end of all the calculations. Diagram 3.4(b) is irrelevant in the present discussion, because it will not introduce any IR divergences in 2+1 dimensions. After using the Feynman trick (3.7), the contribution of the diagram 3.4(a) to the electron self-energy is,

$$-i\Sigma(p) = -e^2 \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \frac{(2-x)^2p^2 + k^2}{[k^2 - a]^2} , \quad (3.89)$$

where we make a shift in the k integration and drop the integral which is odd in k. To perform the momentum integral apply (3.10) and (3.11) and obtain

$$-i\Sigma(p^2) = -\frac{ie^2}{8\pi} m \left[\int_0^1 dx \frac{(2-x)^2}{[m^2x - p^2x(1-x) + \mu^2(1-x)]^{\frac{1}{2}}} \right] . \quad (3.90)$$

Taking $p^2 = m^2$ this becomes

$$-i\Sigma(p^2 = m^2) = \frac{ie^2}{2\pi} m \left[\ln\left(\frac{\mu}{m}\right) - \ln(2) + \frac{1}{2} \right] . \quad (3.91)$$

From all of this, we see that the expression for δm in scalar QED using a small photon mass as an IR regulator, and Pauli-Villars as a UV regulator, (see Appendix B) is

$$\delta m = -\frac{e^2}{4\pi} \left[\ln\left(\frac{\mu}{m}\right) - \ln(2) + \frac{1}{2} - \frac{M}{m} \right] \quad (3.92)$$

where M is the UV divergence associated with the mass renormalisation. It is evident from this that the IR divergence associated with δm is spin independent. To calculate δZ_2 we differentiate (3.90) and then use (3.86) to obtain

$$\delta Z_2 = \frac{e^2}{4\pi}\frac{1}{m}\left[\frac{1}{\mu} + \frac{1}{2}\right]. \tag{3.93}$$

In contrast to the fermionic theory we see that δZ_2 is only linearly IR divergent.

Next we calculate δm and δZ_2 using dimensional regularisation and show explicitly that δZ_2 is finite since, from (3.93), it only has power divergence and dimensional regularisation will set this to zero.

Dimensional Regularisation and Near Mass Shell

For the mass shift in dimensional regularisation we obtain

$$\delta m = -\frac{e^2}{4\pi}\left[\frac{1}{2\hat{\varepsilon}} - \ln(2) - \ln\left(\frac{\Lambda}{m}\right) + \frac{1}{2}\right]. \tag{3.94}$$

By comparison with (3.21), we see that the IR divergent part of δm is spin independent, but the finite part is spin dependent.

As in the fermionic theory, for δZ_2 we obtain Feynman parameter integrals of the form

$$\int_0^1 dx \frac{1 + \mathcal{O}(x)}{x^{5-D}}. \tag{3.95}$$

The most divergent term here is the one with constant numerator. If we perform the integration for $D = 3 - 2\epsilon$ and assume ϵ to be small the integral will diverge around

$x \approx 0$. However, in dimensional regularisation we carry out the integral for D in such a way that the integral is finite and then take the limit $D \to 3$. For the terms that would appear to be linearly divergent, this yields

$$\int_0^1 dx \frac{1}{x^{5-D}} = \frac{1}{D-4} \to -1\,, \quad \text{as} \quad D \to 3\,. \tag{3.96}$$

There are terms of order x, which are now IR logarithms while higher powers of x are finite when $D = 3$. We find that all the IR logarithms and also all the finite terms cancel each other and obtain

$$\delta Z_2 = 0\,. \tag{3.97}$$

Thus using dimensional regularisation as an IR regulator in QED, we see that δZ_2 is finite. We would indeed expect this from (3.93).

We end this subsection by calculating the expression for δm and δZ_2 in the near mass shell scheme. The mass shift in this case is exactly the same as in the fermionic theory, (3.26), which confirms the spin independence of mass renormalisation in this particular regularisation scheme. For the wave function renormalisation we now find that

$$\delta Z_2 = \frac{\mathrm{e}^2}{4\pi}\frac{1}{m}\left[\frac{m}{\Delta} - \ln(2) - 1\right]\,. \tag{3.98}$$

Again there are only linear IR infinities in δZ_2.

This completes our calculations for renormalisation constants in scalar QED. For ease of reference we summarise all the scalar theory results below

	δm	δZ_2
Photon mass	$-\frac{e^2}{4\pi}\left[\ln\left(\frac{\mu}{m}\right) - \ln(2) + \frac{1}{2}\right]$	$\frac{e^2}{4\pi}\frac{1}{m}\left[\frac{1}{\mu} + \frac{1}{2}\right]$
Near mass shell	$-\frac{e^2}{4\pi}\left[\ln\left(\frac{\Delta}{m}\right) - \ln(2) - \frac{1}{2}\right]$	$\frac{e^2}{4\pi}\frac{1}{m}\left[\frac{m}{\Delta} - \ln(2) - 1\right]$
Dim Reg	$-\frac{e^2}{4\pi}\left[\frac{1}{2\hat{\epsilon}} - \ln(2) - \ln\left(\frac{\Lambda}{m}\right) + \frac{1}{2}\right]$	0

3.2.2 The Vertex Correction

We now examine the three-point vertex correction in scalar QED. In comparison with the fermionic case, scalar QED has two extra four-point vertex diagrams which may be seen in Figure 3.5.

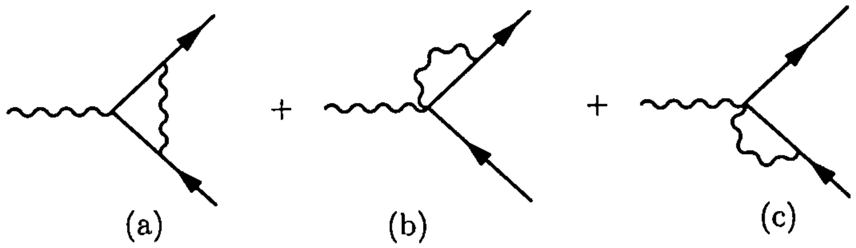


Figure 3.5: *The vertex correction diagrams in scalar QED.*

We now employ the Feynman rules to calculate the expressions for each diagram. In Feynman gauge, the contribution of the diagram 3.5(a) to the vertex is

$$\Gamma^{\mu}_{(1)a}(p,p') = 2e^3 \int \frac{d^D k}{(2\pi)^D} \frac{g_{\rho\sigma}}{k^2} \frac{(2p-k)_\rho (2p'-k)_\sigma (p-k)^\mu}{((p-k)^2 - m^2)((p'-k)^2 - m^2)} \,. \tag{3.99}$$

Our aim is to study the Ward Identity in scalar theory. We therefore take the limit $p = p'$, i.e., the zero momentum transfer limit. In this way we obtain from this diagram

$$\Gamma^{\mu}_{(1)a}(p) = 2e^3 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{(2p-k)^2 (p-k)^{\mu}}{((p-k)^2 - m^2)^2} . \tag{3.100}$$

This may also be written as

$$\Gamma^{\mu}_{(1)a}(p) = 2e^3 [4p^2 I_0^{\mu} - 4p_{\alpha} I_1^{\alpha\mu} + g_{\alpha\beta} I_2^{\alpha\beta\mu}] , \tag{3.101}$$

where

$$I_0^{\mu} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{(p-k)^{\mu}}{((p-k)^2 - m^2)^2} , \tag{3.102}$$

$$I_1^{\alpha\mu} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{k^{\alpha}(p-k)^{\mu}}{((p-k)^2 - m^2)^2} , \tag{3.103}$$

and

$$I_2^{\alpha\beta\mu} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \frac{k^{\alpha} k^{\beta} (p-k)^{\mu}}{((p-k)^2 - m^2)^2} . \tag{3.104}$$

To calculate these integrals, we shall first use dimensional regularisation and follow this by using the small photon mass as a regulator. In the usual way we rewrite I_0 as follows:

$$I_0^{\mu} = 2 \int_0^1 dy \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{x(p-k)^{\mu}}{(k^2 - 2p \cdot kx)^3} . \tag{3.105}$$

Introducing the new integration variables and performing all the necessary integral, we obtain

$$I_0^{\mu} = -\frac{i}{32\pi} \frac{p^{\mu}}{m^3} \left[\frac{1}{\hat{\varepsilon}} - \ln(4) \right] . \tag{3.106}$$

We can follow a similar procedure to calculate $I_1^{\alpha\mu}$ and $I_2^{\alpha\beta\mu}$.

$$I_1^{\alpha\mu} = -\frac{ig^{\alpha\mu}}{16\pi} + \frac{ip^\alpha}{32\pi}\frac{p^\mu}{m^3}\left[\frac{1}{\hat{\varepsilon}} - \ln(4) + 4\right], \tag{3.107}$$

and

$$I_2^{\alpha\beta\mu} = \frac{ip^\mu}{32\pi}(g^{\alpha\beta} - p^\alpha p^\beta) - \frac{i}{32\pi}(g^{\beta\mu}p^\alpha + g^{\alpha\mu}p^\beta). \tag{3.108}$$

Substituting (3.106) - (3.108) into (3.101), we get

$$\Gamma^\mu_{(1)a} = -\frac{ie^3}{2\pi}\frac{p^\mu}{m}\left[\frac{1}{\hat{\varepsilon}} - \ln(4) + 1\right]. \tag{3.109}$$

Finally, the contribution of the diagrams 1.7b and 1.7c to the vertex is similarly found to be

$$\Gamma^\mu_{(1)b} + \Gamma^\mu_{(1)c} = \frac{ie^3}{2\pi}\frac{p^\mu}{m}\left[\frac{1}{\hat{\varepsilon}} - \ln(4) + 1\right]. \tag{3.110}$$

Combining (3.109) and (3.110), we obtain

$$\Gamma^\mu_{(1)}(p = p') = 0. \tag{3.111}$$

The on-shell vertex counter term δZ_1 in scalar QED is defined as follows:

$$\Gamma^\mu_{(1)}(p = p') + 2p^\mu \delta Z_1 = 0. \tag{3.112}$$

Using this, we obtain

$$\delta Z_1 = 0. \tag{3.113}$$

As with δZ_2, the IR logarithms and all the finite corrections cancel each other. Thus we see that dimensional regularisation preserves the Ward identity in 2+1 dimensional scalar QED.

We finish this subsection by writing the expression for δZ_1 in scalar QED, using the photon mass as an IR regulator. We find that

$$\delta Z_1 = \frac{e^2}{4\pi}\frac{1}{m}\left[\frac{1}{\mu} + \frac{1}{2}\right] . \tag{3.114}$$

This is same as (3.93), the expression for δZ_2 in small photon mass scheme and hence this scheme also preserves the Ward identity.

3.3 Summary

In this chapter we have seen that the various Green's functions, such as the 2-point and 3-point functions, have divergences both in the IR and UV domain. In particular, and as expected, the IR problems are worse in 2+1 dimensions than in 3+1 dimensions. This is partly because the mass shift (δm) now has divergences in the IR region. Moreover, the IR divergences associated with the wave function renormalisation constant (δZ_2) are also worse. In 2+1 dimensions δZ_2 has linear, as well as logarithmic, IR divergences in fermionic theory. In order to regulate the IR divergences we have employed different regularisation schemes. We note that dimensional regularisation regulates logarithmic divergences (as $1/\epsilon$ poles) and simply sets all power divergences to zero. The leading IR divergences are spin independent, as would be expected from the calculations in 3+1 dimensions, and which can be understood from the Kulish- Faddeev argument on asymptotic dynamics (see Chapter 2). However, the sub-leading divergences, i.e., the logarithmic divergences in δZ_2, are spin dependent. This shows that the Kulish-Faddeev argument must be modified at this level. We have also calculated δZ_1 in each regularisation scheme and have shown that they all preserve the Ward identity.

Chapter 4

Dressed Charges in 2+1 Dimensions

In this chapter we will use the dressing method to eliminate the infrared divergences of gauge theories in 2+1 dimensions. We will study the 2-point and 3-point dressed Green's functions in 2+1 dimensions. We will show that the various structures associated with the Green's functions are gauge invariant and that the mass shift and the wave function renormalisation become IR finite when we use the dressed Green's function [74]. Finally, we will study the scattering of dressed charges. We will consider both scalar and spinor electrodynamics to study the spin dependence of the propagator but we will only study the scattering vertex in scalar theory. Our method is to extract the IR divergences in the different diagrams and show that they cancel at the level of the integral. It is important to point out here that we work in an arbitrary covariant gauge.

4.1 The Electron Propagator

We begin our study of dressed charges in 2+1 dimensions by considering the electron propagator in scalar QED, which was studied in Chapter 3 in 3+1 dimensions. We first show how to extract the IR divergences in each of the different sorts of diagrams contributing to the dressed electron propagator in scalar QED. We then repeat this, very briefly, in fermionic QED to investigate the spin dependence of the IR divergences associated with the propagator. The dressed field in scalar QED, as discussed in Chapter 2, is given by

$$\phi(x) := h^{-1}(x)\phi(x) = e^{-ieK(x)}e^{-ie\chi(x)}\phi(x)\,. \tag{4.1}$$

where K and χ are defined in (2.65) and (2.64) respectively. An explanation of the origin of the dressing can also be found in [12] and [7]. The Feynman rules for the dressed Green's functions are the usual ones described in the previous chapters with the addition of two new rules corresponding to the dressings as shown in Figure 4.1.

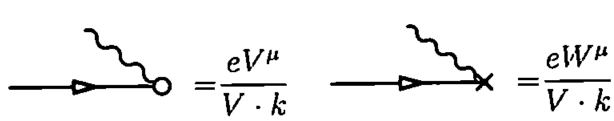


Figure 4.1: *The Feynman rules from expanding the dressing.*

The first vertex comes from the minimal (χ) part of the dressing, and the second corresponds to the additional, separately gauge invariant (K) dressing. Here V and

W are defined as follows:

$$V^{\mu} := k \cdot (\eta + v)\,(\eta + v)^{\mu} - k^{\mu}, \quad W_{\mu} = \frac{k \cdot (\eta + v)\,k_{\mu} - k^2\,(\eta + v)^{\mu}}{k \cdot \eta}, \tag{4.2}$$

where $v = (0, \boldsymbol{v})$ is the velocity of the on-shell particle with momentum $p = m\gamma(1, \boldsymbol{v})$, and k is the incoming momentum of the photon. Note that $W \cdot k = 0$ as a consequence of the gauge invariance of that dressing.

We draw all the possible one loop diagrams for the electron propagator when we include the above dressing and then look at the infrared structure for each of the diagrams. Since the dressed fields are gauge invariant by construction, we need to show that these structures are also gauge invariant. Finally, the cancellation of on-shell infrared divergences will be shown explicitly. The procedure we shall follow is a modification of [23] which is sufficient to treat the richer IR structure of the 2+1 dimensional theory.

The relevant diagrams are shown in Figure 4.2. These include both the minimal and additional dressings, together with all the massless tadpoles shown in Figure 4.3. As we saw in Chapter 3, the usual on-shell propagator given by the sum of Figure 4.2(a) and 4.2(b) displays divergences in the IR region. The remaining diagrams, 4.2(c) – 4.2(j), come from expanding both parts of the dressing, where 4.2(c) – 4.2(e) involve the perturbative expansion of the minimal (χ) part of the dressing (see also Section 3 of [23]); 4.2(f) and 4.2(g) are cross terms from expanding both dressing structures and the diagrams 4.2(h) – 4.2(j) come from expanding the additional (K)

term.

We use the Feynman rules to write down the form of each diagram. To see the gauge invariance of our final result we leave the form of the photon propagator, $D_{\mu\nu}$, completely general. Our procedure is to extract the IR divergences from each diagram for both double and single pole structures.

The contribution of the usual covariant diagram 4.2(b) to the propagator has the form

$$iS^{4.2b}(p) = \frac{e^2}{(p^2-m^2)^2}\int \frac{d^3k}{(2\pi)^3} D_{\mu\nu}\frac{(2p-k)^\mu(2p-k)^\nu}{(p-k)^2-m^2} \,. \tag{4.3}$$

This diagram has an on-shell IR divergence which, as we have seen in Chapter 3 in the absence of a dressing, causes δm (the mass renormalisation constant) to be IR divergent. In the case of 3+1 dimensions we find single pole IR infinities (in Z_2) by extracting a power of $(p^2 - m^2)$. (Details of this can be found in Section 3(a) of [23].) This type of IR divergence is usual when we calculate the wave function renormalisation constant in 3+1 dimensions. The formal procedure is to perform the Taylor expansion about $p^2 = m^2$. After dropping the IR finite term, we obtain from the diagram 4.2(b) the following IR divergent contributions to the mass shift (double pole) and the wave function renormalisation constant (single pole):

$$\begin{aligned} iS^{4.2b}(p) &= -\frac{2e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{p^\mu p^\nu}{(p\cdot k} \\ &+\frac{e^2}{(p^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\left\{\left[-\frac{p^\mu p^\nu}{(p\cdot k)^2}\right]\right. \\ &\left.+\left[-\frac{1}{m^2}\frac{p^\mu p^\nu}{p\cdot k}+\frac{p^\mu k^\nu}{2(p\cdot k)^2}+\frac{p^\nu k^\mu}{2(p\cdot k)^2}-\frac{p^\mu p^\nu}{(p\cdot k)^2}\frac{k^2}{p\cdot k}\right]\right\}. \qquad (4.4) \end{aligned}$$

As expected from power counting, there are only logarithmic divergences in δm but both linear and logarithmic ones in Z_2.

From diagram 4.2(c), using the Feynman rules yields

$$iS^{4.2c}(p) = \frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu}{V\cdot k}\frac{(2p-k)^\nu}{(p-k)^2-m^2}. \qquad (4.5)$$

Simple power counting tells us that the term proportional to p has an off-shell IR divergence which is not well defined. In order to make it well defined we use the identity (see also [7]).

$$\frac{1}{(p-k)^2-m^2} = \frac{1}{p^2-m^2}\left[1+\frac{2p\cdot k-k^2}{(p-k)^2-m^2}\right]. \qquad (4.6)$$

As a consequence, we now have a double pole structure. Using a Taylor expansion to find the single pole structures we obtain the following contribution to the diagram 4.2(c):

$$
\begin{aligned}
iS^{4.2c}(p) &= -\frac{2e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{p^\mu V^\nu}{V\cdot k} \\
&\quad +\frac{e^2}{(p^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\left\{\left[-\frac{p^\mu V^\nu}{p\cdot k\, V\cdot k}\right]\right. \\
&\quad \left. +\left[-\frac{1}{m^2}\frac{p^\mu V^\nu}{V\cdot k}-\frac{V^\mu k^\nu}{2p\cdot k\, V\cdot k}+\frac{p^\mu V^\nu}{2p\cdot k\, V\cdot k}\frac{k^2}{p\cdot k}\right]\right\}. \qquad (4.7)
\end{aligned}
$$

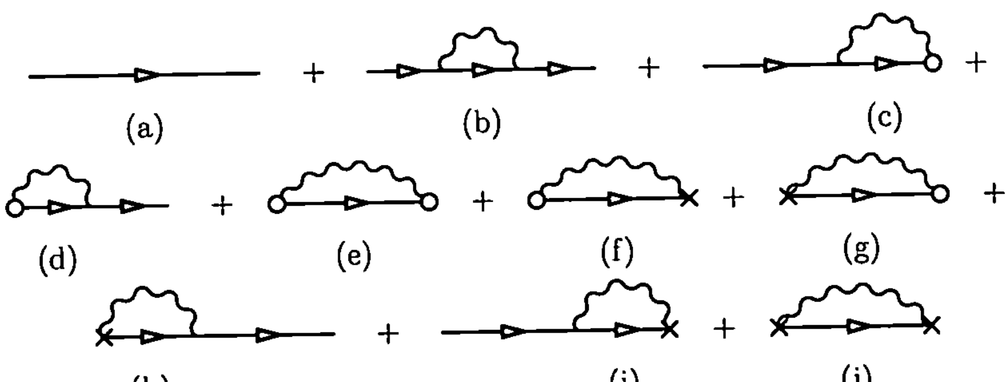


Figure 4.2: *The one-loop Feynman diagrams in the electron propagator which contain IR-divergences when both the minimal and extra dressing are included.*

Here we have ignored the 1 in the square bracket of (4.6), which is a double pole massless tadpole and corresponds to an *odd* k integral. In any reasonable regulator such terms must be zero. The contribution of diagram 4.2(d) is easily seen to be identical to this. Note that the contribution of diagrams 4.2(h) and 4.2(i) to the propagator can now be immediately obtained by changing all the V-factors to W's in (4.7).

The off-shell divergences in diagrams 4.2(e) – 4.2(g) and 4.2(j), become even worse in 2+1 dimensions. To compensate for this we need to use the technique (4.6) twice to make them off-shell IR finite. As a result each diagram has a double pole IR infinity. We perform a Taylor expansion to extract single pole structures. To see this explicitly we calculate the diagram 4.2(e) and the contribution to the propagator is

$$iS^{4.2e}(p) = e^2 \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V^\nu}{(V \cdot k)^2} \frac{1}{(p-k)^2 - m^2} \,. \tag{4.8}$$

This diagram has off-shell IR divergences and we make use of (4.6) to rewrite it as

$$iS^{4.2e}(p) = \frac{e^2}{p^2 - m^2} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V^\nu}{(V \cdot k)^2} \left[1 + \frac{2p \cdot k - k^2}{(p-k)^2 - m^2} \right] . \tag{4.9}$$

It is interesting to see that the first term in the square bracket is a single pole massless tadpole which cancels the diagrams 4.3(a) and 4.3(b). When the remaining rainbow diagrams are calculated, all other diagrams in Figure 4.3 are cancelled. By power counting we can see that the third term in the square bracket of (4.9) is well defined, but it is IR divergent on-shell. The second term still has an off-shell IR divergence. We use the identity (4.6) again and obtain

$$\begin{aligned} iS^{4.2e}(p) &= \frac{e^2}{(p^2 - m^2)^2} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V^\nu}{(V \cdot k)^2} 2p \cdot k \left[1 + \frac{2p \cdot k - k^2}{(p-k)^2 - m^2} \right] \\ &+ \frac{e^2}{p^2 - m^2} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V^\nu}{(V \cdot k)^2} \frac{k^2}{2p \cdot k} \,. \end{aligned} \tag{4.10}$$

As usual, we have neglected an *odd* double pole massless tadpole. All the other integrals are now well defined. We now go on-shell and drop the IR finite terms to establish the following contribution of the diagram 4.2(e):

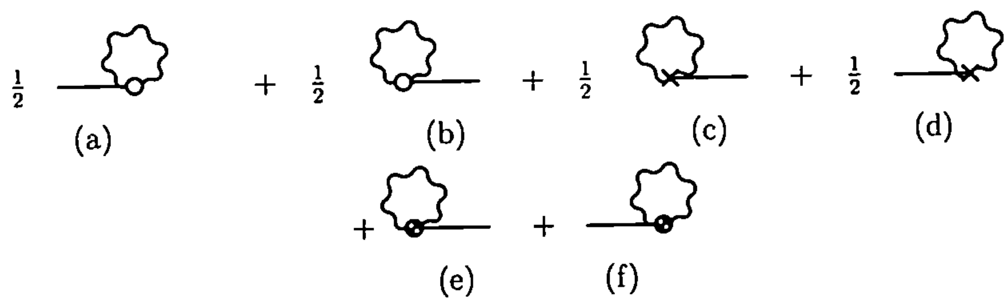


Figure 4.3: *All these one loop massless tadpoles will cancel during the process of extracting IR divergences from diagrams 4.2e - 4.2g and 4.2j. The hatched circle vertex indicates the generic contributions of both parts of the dressing.*

$$\begin{aligned} iS^{4.2e}(p) &= \frac{e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}p\cdot k \\ &\quad -\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu V^\nu}{(V\cdot k)^2}\left[1+\frac{p\cdot k}{m^2}\right]\,. \end{aligned} \tag{4.11}$$

The contribution of the rainbow diagram 4.2(j) to the propagator can now be easily obtained by changing all the V-factors to W's in (4.11). We change V to W only for the diagrams 4.2(f) and 4.2(g) in the same equation. It is important to note from the above calculation that we have both logarithmic and linear divergent structures for the single pole, but only a logarithmic divergent structure for the double pole. This is in accord with power counting and the perturbative calculations of the (non-dressed) Green's function in Chapter 3

Gauge invariant structures:

We now combine these results to obtain the various gauge invariant structures for the infra-red divergent terms which arise from the simple pole and its residue in the case of the dressed electron propagator.

Double pole

The double pole structure corresponds to the mass renormalisation of the electron propagator. All the IR divergent terms for the dressed propagator in the double pole can be written in the following gauge invariant form:

$$-\frac{e^2}{(p^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{p^\nu}{p\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\right\}2p\cdot k\,. \tag{4.12}$$

This form confirms the gauge invariance of the dressed Green's functions: any modification of the Feynman gauge photon propagator will introduce either a k_μ or k_ν factor, but these additional structures will vanish on multiplying these into the square bracket in the above structure. The contraction of either k_μ or k_ν into each of the above structures is zero, confirming the gauge invariance of our original construction. We note here that we have dropped double pole *odd* massless tadpoles, which are not, themselves, separately gauge invariant but which must vanish in any suitable choice of regularisation scheme.

Single Pole

As we have two different types of IR divergences in the single pole structure, i.e. both linearly divergent and logarithmically divergent singularities, we need to find the (gauge invariant) structures for each type. These are, of course, all associated with the wave function renormalisation of the propagator.

Linear Divergences

We can write all the linear IR divergences that arise from the single pole in one gauge invariant structure, which has the form:

$$-\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{p^\nu}{p\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\right\}. \tag{4.13}$$

This is similar to (4.12) (up to the factor $2p\cdot k$) and is similarly gauge invariant.

Logarithmic divergences

All the logarithmically divergent terms can be written in the following form:

$$\frac{e^2}{p^2-m^2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\left\{\left[\frac{k^\nu}{p\cdot k}-\frac{p^\nu}{p\cdot k}\frac{k^2}{p\cdot k}\right]\right.$$
$$\left.-\left[\frac{p^\nu}{p\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\frac{p\cdot k}{m^2}\right\}, \tag{4.14}$$

which is gauge invariant.

This completes our check of the gauge invariance of the infra-red divergences in the dressed matter in 2+1 dimension. We shall now show that these IR divergences cancel at the correct point on the mass shell.

Cancellation of IR divergences:

It is useful to consider the linear combination:

$$\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}. \tag{4.15}$$

Using the definitions of V^μ and W^μ given by (2.66), we observe that this combination adds to zero at the correct point on the mass shell, i.e.,

$$\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}=0,\quad \text{if}\quad p^\mu=m\gamma(\eta+v)^\mu\,. \tag{4.16}$$

This is also a consequence of the *dressing equation* (2.52): expanding $u\cdot\partial h^{-1}=-ieh^{-1}u\cdot A$, in e, and putting $p^\mu=u^\mu$, the correct point on the mass shell, we will obtain (4.16). Applying this to (4.12), (4.13) and (4.14) we find that the IR divergences associated with mass and wave function renormalisations are zero at the level of the on-shell integrand. (As in 3+1 dimensions, this cancellation does not occur if the dressing parameter $\boldsymbol{v}$ and on-shell point do not correspond to each other via $p^\mu=mu^\mu=m\gamma(\eta+v)^\mu$.) This observation is strong evidence for the applicability of dressings to the 2+1 dimensional theory.

Having shown the cancellation of the various IR divergences that occur in the dressed electron propagator in 2+1 dimensions, we finish this section by briefly reporting the results of parallel calculations in fermionic QED.

The double pole gauge invariant structure, (4.12), is identical in the fermionic theory if $1/(p^2-m^2)$ is replaced by $1/(\not{p}-m)$, confirming the spin independence of the IR divergences in mass renormalisation in 2+1 dimensions. As in scalar theory there are *odd* massless tadpoles which are not separately gauge invariant but must vanish.

The single pole linear IR structures (4.13) are also identical in both theories, the leading IR singularities being spin independent as one would expect. However the logarithmic structures in fermionic QED are different from those of scalar QED. In the former we find the following logarithmic IR structure:

$$e^2 \int \frac{d^3k}{(2\pi)^3} \left[\frac{p^\mu}{p \cdot k} - \frac{V^\mu}{V \cdot k} - \frac{W^\mu}{V \cdot k}\right] D_{\mu\nu} \left[\frac{k^\nu}{p \cdot k} - \frac{p^\nu}{p \cdot k}\frac{k^2}{p \cdot k}\right] \frac{1}{\not{p} - m}. \tag{4.17}$$

Therefore the sub-leading, logarithmic IR divergences associated with the wave function renormalisation are spin dependent in 2+1 dimensions. This confirms our previous result in Chapter 3. Since the logarithmic structures in fermionic QED are different from those of the scalar theory it is essential for us to check that δZ_2 is IR finite. Using (4.15), we can immediately show that the electron propagator in fermionic QED is also IR finite.

We have thus demonstrated that as in 3+1 dimensions [23], the dressed theory offers a way to describe charged particles propagating on-shell. The dressing is able to deal with the significantly more complex IR structures in this lower dimensional theory. This could be useful in the study of condensed matter systems [32–36].

This completes our examination of the IR divergences in the electron propagator. In the next section we will look at the dressed vertex.

4.2 The Scattering Vertex

The aim of this section is to investigate the IR divergent terms in the dressed, on-shell three-point vertex. We first show that all the structures are gauge invariant and then study the cancellation of these divergences.

Let us consider the scattering of dressed charges off a current in 2+1 dimensions. The relevant diagrams at one loop are shown in Figure 4.4 on p. 99 and include both the minimal and the additional dressings. We will consider the theory of scalar QED to avoid the complication arising from Dirac's gamma matrices. We denote the vertex by Γ.

Many of these diagrams are propagator corrections on one or other legs and as such have been effectively calculated in the previous section on the electron propagator. The method of extracting the divergences is essentially the same as in the propagator case. The new diagrams are 4.4(a), 4.4(d), 4.4(g), 4.4(h) and 4.4(k) - 4.4(o). We note here that we have only considered those diagrams which can generate IR divergences and a pole for each of the external legs. We shall ignore all massless tadpoles but will return to them later. Diagrams 4.4(g), 4.4(h), 4.4(n) and 4.4(o) can be neglected in 3+1 dimensions, because they will not yield poles in one of the legs (see also [23]). However they do yield IR divergences and a pole in each leg in the 2+1 dimensional case and must therefore be taken into account.

The IR divergent contribution of the usual covariant diagram, 4.4(a) to the one loop vertex, has the form

$$\Gamma^{1a}(p,p') = \frac{ie^2}{(p^2-m^2)(p'^2-m^2)} \int \frac{d^3k}{(2\pi)^3} \frac{D_{\mu\nu}}{p\cdot k\, p'\cdot k}$$
$$\times \left[p^\mu p'^\nu + \frac{1}{2}\left(p^\mu p'^\nu \frac{k^2}{p\cdot k} + p^\mu p'^\nu \frac{k^2}{p'\cdot k} - p^\mu k^\nu - p'^\nu k^\mu \right)\right] . \tag{4.18}$$

Now consider the rainbow type diagrams 4.4(d) and 4.4(k) – (m). In order to extract on-shell IR infinities, they have to be factorised twice in each leg using the identity (4.6), thus

$$\frac{1}{(p-k)^2-m^2} = \frac{1}{p^2-m^2}\left[1+\frac{2p\cdot k-k^2}{(p-k)^2-m^2}\right] , \tag{4.19}$$
$$\frac{1}{(p'-k)^2-m^2} = \frac{1}{p'^2-m^2}\left[1+\frac{2p'\cdot k-k^2}{(p'-k)^2-m^2}\right] . \tag{4.20}$$

As a consequence we need to perform a Taylor expansion to extract a pole in each leg. We find that the contribution of the diagram 4.4(d) to the vertex is

$$\Gamma^{1d}(p,p') = -\frac{ie^2}{(p^2-m^2)(p'^2-m^2)^2} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V'^\nu}{V\cdot k\, V'\cdot k} 2p'\cdot k \tag{4.21}$$
$$-\frac{ie^2}{(p^2-m^2)^2(p'^2-m^2)} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V'^\nu}{V\cdot k\, V'\cdot k} 2p\cdot k$$
$$-\frac{ie^2}{(p^2-m^2)(p'^2-m^2)} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu} \frac{V^\mu V'^\nu}{V\cdot k\, V'\cdot k}\left[\frac{p'\cdot k}{m^2}+\frac{p\cdot k}{m^2}+1\right] .$$

Here we have dropped all massless tadpoles for the double pole (that is a double pole in one leg and a single pole in the other), and for the single pole. They will be discussed later. The contribution of the rainbow diagram, 4.4(k), to the vertex can now be obtained immediately by changing all the V-factors to W's (and similarly for V' and W') in (4.21). Thus we change only V' to W' for the diagram 4.4(l) and change V to W for the diagram 4.4(m) in the same equation. The expressions for the diagrams 4.4(g) and 4.4(h) are as follows:

$$\Gamma^{1g}(p,p') = -\frac{ie^2}{(p^2-m^2)(p'^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{V^\mu p'^\nu}{V\cdot k\, p'\cdot k}2p'\cdot k \qquad (4.22)$$
$$+\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{1}{m^2}\frac{V^\mu p'^\nu}{V\cdot k\, p'\cdot k}p'\cdot k,$$

$$\Gamma^{1h}(p,p') = -\frac{ie^2}{(p^2-m^2)^2(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{p^\mu V'^\nu}{p\cdot k\, V'\cdot k}2p\cdot k \qquad (4.23)$$
$$+\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\frac{1}{m^2}\frac{p^\mu V'^\nu}{p\cdot k\, V'\cdot k}p\cdot k.$$

Finally, the expressions for diagrams 4.4(n) and 4.4(o) can be obtained by changing V to W and V' to W' in (4.22) and (4.23) respectively. We can now combine all the IR divergent expressions arising in the one loop vertex correction as shown in Figure 4.4. As in the case of the propagator we also calculate the structures which have a double pole in one leg (as well as a single pole in the other) to check that this corresponds to mass renormalisation.

Double poles

In the calculation that follows we will extract all the double poles in one leg structures associated with the dressed vertex in scalar QED. These correspond to mass shift renormalisation for the vertex correction. There are two structures, one on each leg, coming from the diagrams of Figure 4.4. The covariant diagram, Figure 4.4(a), does not have any double pole structure. Double pole structures for the leg with momentum p' are

$$-\frac{ie^2}{(p^2-m^2)(p'^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p'^\mu}{p'\cdot k}-\frac{V'^\mu}{V'\cdot k}-\frac{W'^\mu}{V'\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{p'^\nu}{p'\cdot k}-\frac{V^\nu}{V\cdot k}-\frac{W^\nu}{V\cdot k}\right]\right\}2p'\cdot k\,, \qquad (4.24)$$

and for the leg with momentum p

$$-\frac{ie^2}{(p^2-m^2)^2(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{p^\nu}{p\cdot k}-\frac{V'^\nu}{V'\cdot k}-\frac{W'^\nu}{V'\cdot k}\right]\right\}2p\cdot k\,. \qquad (4.25)$$

These results show the expected symmetry between p and p' and make the gauge invariant nature of our dressed Green's functions manifest. By power counting they are at most logarithmically divergent.

Single poles

We can combine all the IR divergent terms associated with one pole in each leg in the following forms:

$$\Gamma_1^{\rm IR}(p,p')=\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}\right]D_{\mu\nu}(k)\left[\frac{p'^\nu}{p'\cdot k}-\frac{p^\nu}{p\cdot k}\right]\right.$$
$$\left.-\left[\frac{p'^\mu}{p'\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\left[\frac{p^\mu}{p\cdot k}-\frac{V'^\mu}{V'\cdot k}-\frac{W'^\mu}{V'\cdot k}\right]\right\}(4.26)$$

$$\Gamma_2^{\rm IR}(p,p')=\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p'^\mu}{p'\cdot k}-\frac{V'^\mu}{V'\cdot k}-\frac{W'^\mu}{V'\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{1}{2}\left(\frac{k^\nu}{p'\cdot k}-\frac{p'^\nu}{p'\cdot k}\frac{k^2}{p'\cdot k}\right)-\frac{p'\cdot k}{m^2}\left(\frac{p'^\mu}{p'\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right)\right]\right\}\,,(4.27)$$

$$\Gamma_3^{\rm IR}(p,p')=\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{p^\mu}{p\cdot k}-\frac{V^\mu}{V\cdot k}-\frac{W^\mu}{V\cdot k}\right]D_{\mu\nu}(k)\right.$$
$$\left.\times\left[\frac{1}{2}\left(\frac{k^\nu}{p\cdot k}-\frac{p^\nu}{p\cdot k}\frac{k^2}{p\cdot k}\right)-\frac{p\cdot k}{m^2}\left(\frac{p^\mu}{p\cdot k}-\frac{V'^\mu}{V'\cdot k}-\frac{W'^\mu}{V'\cdot k}\right)\right]\right\}\,,\ (4.28)$$

$$\Gamma_4^{\rm IR}(p,p') = \frac{1}{2}\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\left[\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}\right]D_{\mu\nu}(k)$$
$$\times\left[\left(\frac{k^2}{p\cdot k}-\frac{k^2}{p'\cdot k}\right)\frac{k^\nu}{k^2}-\left(\frac{k^2}{p\cdot k}\frac{p^\nu}{p\cdot k}-\frac{k^2}{p'\cdot k}\frac{p'^\nu}{p'\cdot k}\right)\right]. \qquad (4.29)$$

Notice that the structure (4.26) has linearly divergent integrals while the structures (4.27)–(4.29) are only logarithmically divergent. The contraction of either k_μ or k_ν into each of the above structures is zero, and confirms the gauge invariance of our original construction.

Cancellation of IR divergences:

Using again the simple argument based on (4.16), we can see that all the double pole structures, as well as the structures (4.26), (4.27) and (4.28), will vanish at the level of the on-shell integrand. The structure Γ_4 is not cancelled by this but it simplifies to

$$\Gamma_4^{IR} = \frac{1}{2}\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\left[\frac{p^\mu}{p\cdot k}-\frac{p'^\mu}{p'\cdot k}\right]\left[\frac{p^\mu}{(p\cdot k)^2}-\frac{p'^\mu}{(p'\cdot k)^2}\right]. \qquad (4.30)$$

To show that this structure also vanishes, we use the fact that $p\cdot k$ and $p'\cdot k$ are actually $p\cdot k+i\varepsilon$ and $p'\cdot k+i\varepsilon$. The poles in k_0 therefore lie in the same half of the complex plane, and hence the integral vanishes by Cauchy's theorem.

Massless Tadpoles:

We now return to the massless tadpoles that we have neglected in the above process of extracting IR divergent terms from the dressed scattering vertex. It is important to check whether they are gauge invariant and if they all vanish. We have not yet included the diagrams of Figure 4.5 on p. 100.

We use the Feynman rules to write down the diagrams of Figure 4.5 and obtain,

$$\Gamma^{\mathrm{TP}}(p,p') = -\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}\left\{\frac{1}{2}\left[\frac{V'^{\mu}V'^{\nu}}{(V'\cdot k)^2}+\frac{V^{\mu}V^{\nu}}{(V\cdot k)^2}\right]\right. \quad (4.31)$$
$$\left.+\frac{1}{2}\left[\frac{W'^{\mu}W'^{\nu}}{(V'\cdot k)^2}+\frac{W^{\mu}W^{\nu}}{(V'\cdot k)^2}\right]+\left[\frac{W'^{\mu}V'^{\nu}}{(V'\cdot k)^2}+\frac{W^{\mu}V^{\nu}}{(V\cdot k)^2}\right]\right\}.$$

Recall that in the process of extracting soft divergences from the rainbow type diagrams we have ignored the following massless tadpoles

$$\Gamma^{\mathrm{TP}}(p,p') = \frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\frac{D_{\mu\nu}}{V\cdot k\,V'\cdot k}[V^{\mu}V'^{\nu}+W^{\mu}W'^{\nu} \quad (4.32)$$
$$+V^{\mu}W'^{\nu}+W^{\mu}V'^{\nu}]$$
$$+\frac{ie^2}{(p^2-m^2)(p'^2-m^2)^2}\int\frac{d^3k}{(2\pi)^3}\frac{D_{\mu\nu}}{V\cdot k\,V'\cdot k}[V^{\mu}V'^{\nu}+W^{\mu}W'^{\nu}$$
$$+V^{\mu}W'^{\nu}+W^{\mu}V'^{\nu}]2p'\cdot k$$
$$+\frac{ie^2}{(p^2-m^2)^2(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\frac{D_{\mu\nu}}{V\cdot k\,V'\cdot k}[V^{\mu}V'^{\nu}+W^{\mu}W'^{\nu}$$
$$+V^{\mu}W'^{\nu}+W^{\mu}V'^{\nu}]2p\cdot k\,.$$

In this expression the second and third integrals are *odd* double pole massless tadpoles and can therefore be dropped in any reasonable regularisation scheme. The first integral in the expression is even and linear, and so does not vanish, but it can be combined with (4.31) and we find

$$\Gamma^{\rm TP}(p,p') = -\frac{ie^2}{(p^2-m^2)(p'^2-m^2)}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}\left\{\left[\frac{V^\mu}{V\cdot k}-\frac{V'^\mu}{V'\cdot k}\right]D_{\mu\nu}(k)\right. \times\left[\frac{V^\nu}{V\cdot k}-\frac{V'^\nu}{V'\cdot k}+2\left(\frac{W^\nu}{V\cdot k}-\frac{W'^\nu}{V'\cdot k}\right)\right] \left.+\left[\frac{W^\mu}{V\cdot k}-\frac{W'^\mu}{V'\cdot k}\right]D_{\mu\nu}(k)\left[\frac{W^\nu}{V\cdot k}-\frac{W'^\nu}{V'\cdot k}\right]\right\}. \tag{4.33}$$

This form confirms the gauge invariance of all the massless tadpoles with a pole in each leg. They cancel each other when $q=0$ but not when $q\neq 0$. These massless tadpoles also exist in 3+1 dimensions but are removed through the process of dimensional regularisation.

We also argue that these tadpoles will vanish in dimensional regularisation for two reasons: (i) they are linearly divergent and (ii) they are massless tadpoles. A question that arises is whether or not this prescription is acceptable in 2+1 dimensions.

Finally, in scalar QED we have massless tadpoles which are shown in Figure 4.6. They are even and there is no IR divergence associated with these diagrams. However, there exist UV divergences and they are related to the standard mass renormalisation.

4.3 Summary

In this chapter we have studied the IR properties of the on-shell electron propagator in 2+1 dimensions, where the divergence structures are far richer than in 3+1 dimensions. We have shown that if we use the full dressing to solve the dressing equation, then both the mass shift and the wave function renomalisation constant are IR finite, despite there now being both linear and logarithmic IR structures. The different IR structures cancel separately at the correct point on the mass shell. These results were established in both fermionic and scalar QED.

We then calculated the IR properties of the scattering of dressed charges in 2+1 dimensional scalar QED and showed that the IR divergences are gauge invariant. They all cancelled for zero momentum transfer ($q = 0$). There are, however, IR linear infinities associated with even massless tadpoles that do not cancel when $q \neq 0$. All the massless tadpoles can be dropped if we are using dimensional regularisation. A question that remains is whether this cancellation occurs without dimensional regularisation. To clarify these unanswered questions, further work is needed using a different gauge invariant regulator that is compatible with the dressing. The best known regulators are (i) the photon mass scheme, (ii) the near mass shell scheme. The former, since it changes the theory, ought to modify the dressing and the latter is not suitable for work with a dressing since this is constructed to describe on-shell physical particles. There thus remains an important and open question of how to modify the dressing for a regularisation scheme other than dimensional regularisation.

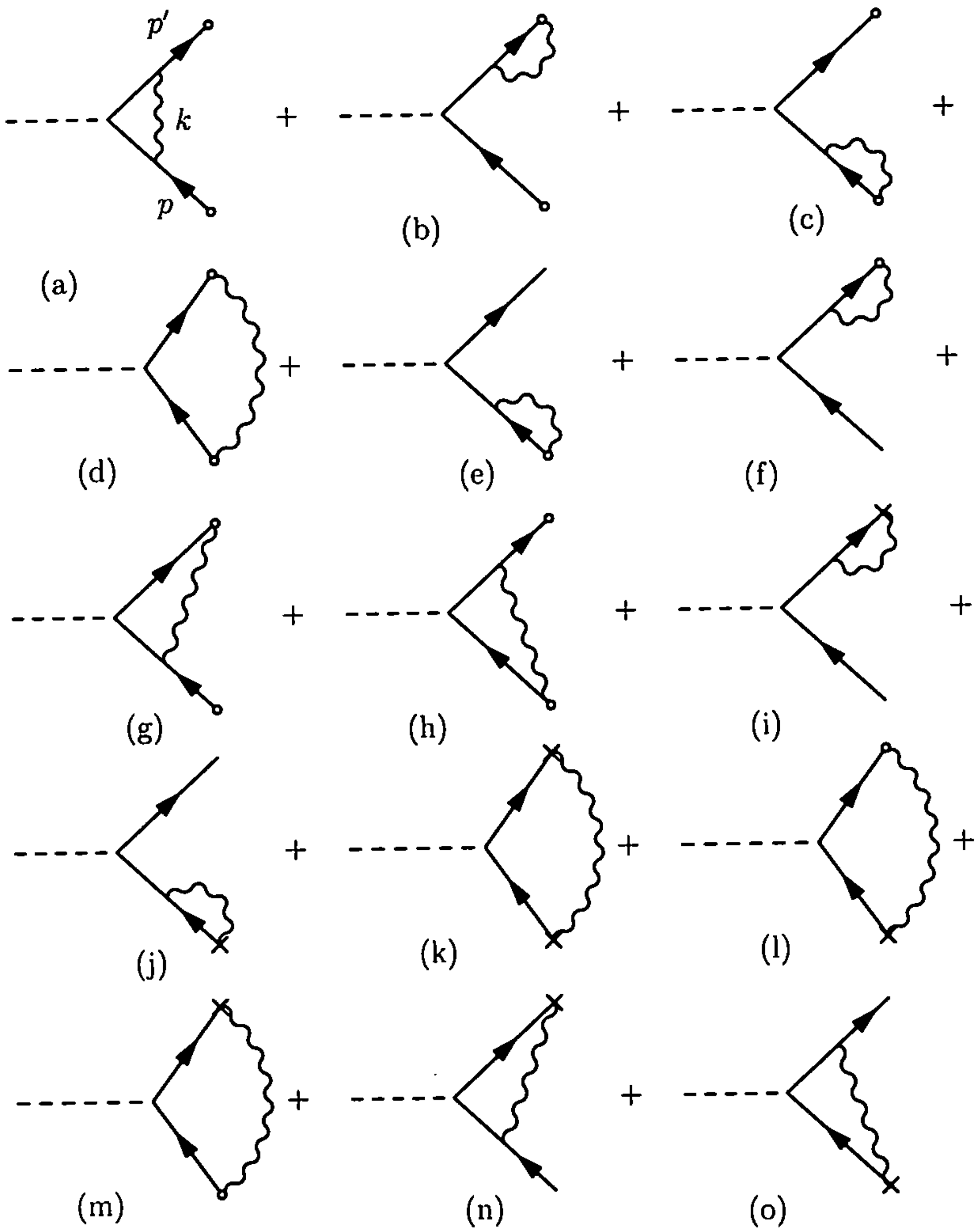


Figure 4.4: *All one-loop Feynman diagrams in the scattering vertex which contain IR-divergences when we include both the minimal and extra dressings.*

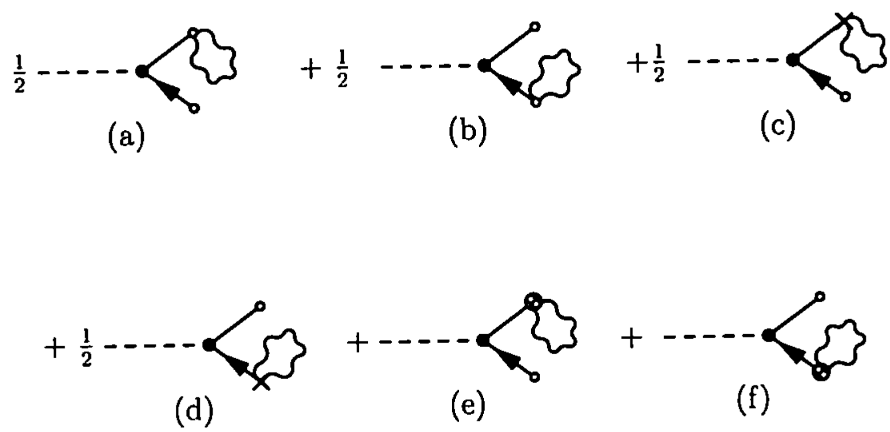


Figure 4.5: *All one-loop massless tadpoles from the dressed scattering vertex when we include both the minimal and the extra dressing.*

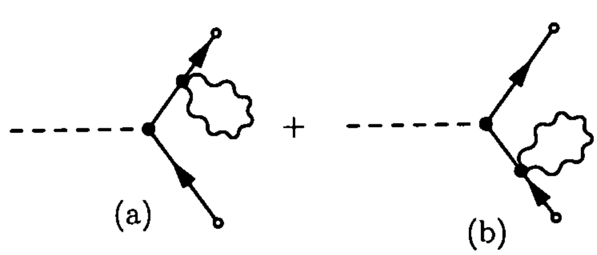


Figure 4.6: *Covariant massless tadpole diagrams in scalar QED.*

Chapter 5

Bloch-Nordsieck and IR Divergences in Gauge Theories

In this chapter we will use the Bloch-Nordsieck (BN) method to study the IR divergences at the level of the cross-section in both four and three dimensional Quantum Electrodynamics. This is the most common approach in 3+1 dimensions. Details of this method can be found in [6] and in Chapter 13 of [62].

The idea underlying this technique is as follows. In a process where a matter particle scatters off a potential the momentum integrals for S-matrix elements generally diverge in the IR domain. These IR divergences can be cancelled at the level of the cross-section by adding the emission of soft photons which are not separately observed because of the finite resolution of any experimental detector. The experimental cross-section, it is argued, does not restrict the number of unobserved photons which may emerge from any scattering and it should correspond to the sum of all these possibilities. The technique is to calculate cross-sections for the real and virtual photons separately and then combine these cross-sections. This is known to give a finite result in 3+1 dimensions. We will first show that the IR divergences associated with the scattering of matter cancel in 3+1 dimensions. We will then apply this technique to 2+1 dimensions where as we have seen many times the IR divergences are worse. Before we study the BN approach, it is important to show that the S-matrix is gauge invariant. We will show that this is true for any dimension.

After studying scattering off a scalar current, we will study the process of a charged particle scattering off a photon. In 3+1 dimensions the IR divergences associated with the S-matrix element are independent of the vertex. However we will see that the IR structure in 2+1 dimensional Coulomb scattering depends on the vertex. Thus we shall extract the IR divergences associated with both virtual and real soft photon emissions in 2+1 dimensions. After this we will try to apply the BN method to the IR in 2+1 dimensional abelian gauge theories with massive matter.

5.1 The S-Matrix

To show the gauge invariance of the S-matrix let us consider the scattering of charges in D dimensions. For simplicity, we will consider the case of scalar electrodynamics, where charged particles have spin zero, mass m and charge e. The diagrams corresponding to this are shown in Figure 5.1 where the factors of 1/2 are due to the standard $Z_2^{-1/2}$ factors in the LSZ formalism:

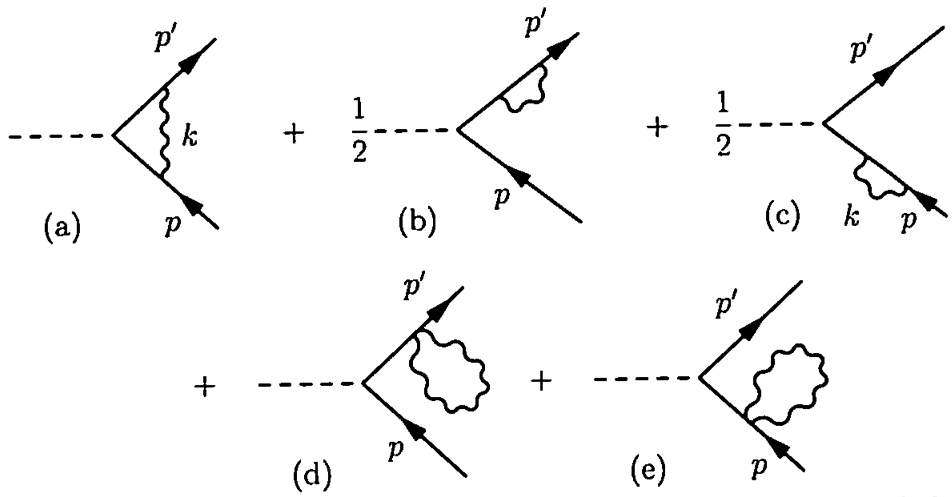


Figure 5.1: *All the one loop Feynman diagrams for the scattering of charges.*

The LSZ reduction formula (see Chapter 5-1-3 in [73]) tells us that we should extract a pole in each leg. For the type of diagrams 5.1(b) and 5.1(c), both of which have a double pole in one leg, we use a Taylor expansion about the momentum flowing on that leg in order to obtain a single pole structure. We also perform mass renormalisation for each double pole. The massless tadpole diagrams 5.1(d) and 5.1(e) do not make any contribution, because there are no matter propagators in the loop. They will only contribute to the ultraviolet (UV) divergences associated with mass renormalisation.

Consider the diagram 5.1(a), which already has a single pole in each leg. Use of the Feynman rules yields

$$\Gamma^{5.1a}_{(1)}(p,p') = -ie^2 \int \frac{d^Dk}{(2\pi)^D} D_{\mu\nu} \frac{(2p'-k)^\mu (2p-k)^\nu}{[(p'-k)^2 - m^2][(p-k)^2 - m^2]}\,, \tag{5.1}$$

where we have suppressed the poles for external legs. This is well defined as it has no off-shell IR divergences. Therefore, at the correct point on the mass shell we may write

$$\Gamma^{a}_{(1)}(p,p') = -ie^2 \int \frac{d^D k}{(2\pi)^D} D_{\mu\nu} \frac{(2p'-k)^\mu (2p-k)^\nu}{[k^2 - 2p'\cdot k][k^2 - 2p'\cdot k]} \,. \tag{5.2}$$

Here the entire expression is needed for the desired gauge invariance of the S-matrix. Next, for diagram 5.1(b), using the Feynman rules we obtain

$$\Gamma^{5.1b}_{(1)}(p,p') = -\frac{ie^2}{p'^2 - m^2} \int \frac{d^D k}{(2\pi)^D} D_{\mu\nu} \frac{(2p'-k)^\mu (2p'-k)^\nu}{[(p'-k)^2 - m^2]} \,. \tag{5.3}$$

This diagram does not have any IR divergences off-shell but there is a double pole at p'. We make a Taylor expansion about $p'^2 - m^2$ to find a pole in each leg. The first term in the expansion corresponds to the mass renormalisation, which we will discuss later. The second term involves the first order differentiation of the integrand with respect to the momentum variable p'. After some algebra, we obtain

$$\begin{aligned}\Gamma^{5.1b}_{(1)}(p,p') = -ie^2 \int \frac{d^D k}{(2\pi)^D} \frac{D_{\mu\nu}}{m^2} \Bigg[&\left(\frac{2p'^\mu (2p'-k)^\nu}{k^2 - 2p'\cdot k} \right) \\ &- \left(\frac{(m^2 - p_2\cdot k)(2p'-k)^\mu (2p'-k)^\nu}{(k^2 - 2p'\cdot k)^2} \right) \Bigg] \,.\end{aligned} \tag{5.4}$$

Finally, for diagram 5.1(c) we change p to p' in the above integrand and obtain:

$$\begin{aligned}\Gamma^{5.1c}_{(1)}(p,p') = -ie^2 \int \frac{d^D k}{(2\pi)^D} \frac{D_{\mu\nu}}{m^2} \Bigg[&\left(\frac{2p^\mu (2p-k)^\nu}{k^2 - 2p\cdot k} \right) \\ &- \left(\frac{(m^2 - p\cdot k)(2p-k)^\mu (2p-k)^\nu}{(k^2 - 2p\cdot k)^2} \right) \Bigg] \,.\end{aligned} \tag{5.5}$$

The next step in the process is to combine all the diagrams in Figure 5.1. Our final result for the S-matrix element is now given by

$$\Gamma_{(1)}(p,p') = -ie^2 \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{1}{2m^2} \left[2p'^{\mu} + \frac{(2p'-k)^{\mu}}{k^2 - 2p' \cdot k} p_2 \cdot k \right] D_{\mu\nu} \frac{(2p'-k)^{\nu}}{k^2 - 2p' \cdot k} \right.$$
$$+ \left[2p^{\mu} + \frac{(2p-k)^{\mu}}{k^2 - 2p \cdot k} p \cdot k \right] D_{\mu\nu} \frac{(2p-k)^{\nu}}{k^2 - 2p \cdot k}$$
$$\left. - \left[\frac{(2p'-k)^{\mu}}{k^2 - 2p' \cdot k} - \frac{(2p-k)^{\mu}}{k^2 - 2p \cdot k} \right] D_{\mu\nu} \left[\frac{(2p'-k)^{\nu}}{k^2 - 2p' \cdot k} - \frac{(2p-k)^{\nu}}{k^2 - 2p \cdot k} \right] \right\} . \quad (5.6)$$

Although this is the correct final expression, because of the first two structures in (5.6) it does not appear to be gauge invariant. The next section shows that this is not the case and (5.6) is indeed gauge invariant.

5.1.1 Gauge Invariance

If we contract either k_μ or k_ν into (5.6) we see that the last structure vanishes, but it is not clear if the first two structures vanish. It is not obvious therefore that the expression is gauge invariant. If we consider the Lorentz class of gauges, however, the gauge dependent part involves $k_\mu k_\nu$ in the numerator. If we contract this into (5.6) we obtain,

$$\Gamma^{\mathrm{GD}}_{(1)}(p,p') = -ie^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{2m^2} \frac{1}{k^4} (p' \cdot k + p \cdot k) \, , \quad (5.7)$$

where $\Gamma^{\mathrm{GD}}_{(1)}$ denotes the gauge dependent part of the S-matrix. Clearly the expression (5.7) is an odd massless tadpole which vanishes upon integration. This is also true in an arbitrary gauge. The photon propagator corresponding to this gauge is

$$D_{\mu\nu} = -\frac{1}{k^2} \left(g_{\mu\nu} - \frac{N_\mu k_\nu + N_\nu k_\mu}{k \cdot N} + \frac{N^2 k_\mu k_\nu}{(k \cdot N)^2} \right) . \quad (5.8)$$

Here the gauge dependent part involves terms linear in k, as well as the quadratic one. The gauge dependent part for the S-matrix element in any arbitrary gauge is thus given by

$$\Gamma^{\rm GD}_{(1)}(p,p') = ie^2 \int \frac{d^Dk}{(2\pi)^D} \frac{1}{2m^2} \frac{1}{k^2} \left[\left(\frac{2}{k \cdot N} (p' \cdot N + p \cdot N) \right) - \left(\frac{N^2}{(k \cdot N)^2} (p' \cdot k + p \cdot k) \right) \right] , \qquad (5.9)$$

which is also an odd massless tadpole. This makes manifest the gauge invariance of the charges scattering off a potential. This entire argument can be generalised to matter scattering off a virtual photon.

5.1.2 Infrared Divergences

3+1 dimensions

We now extract all the IR divergences from (5.6) in 3+1 dimensions. These arise from the last structure in (5.6),

$$\Gamma_{(1)}(p,p') = \frac{ie^2}{2} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right\} \frac{g_{\mu\nu}}{k^2} \left\{ \frac{p'^\nu}{p' \cdot k} - \frac{p^\nu}{p \cdot k} \right\} . \qquad (5.10)$$

To cancel these divergences we can use the Bloch-Nordsieck argument, which will be shown explicitly in the next section.

2+1 dimensions

In 2+1 dimensions, by power counting, we have an additional contribution from the structures in (5.6). We have the following divergent terms as $k^2 \to 0$:

$$\Gamma_{(1)} = \Gamma^1_{(1)} + \Gamma^2_{(1)} + \Gamma^3_{(1)} , \tag{5.11}$$

where we have the linear corrections

$$\Gamma^1_{(1)} = \frac{ie^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \left[\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right] \left[\frac{p'_\mu}{p' \cdot k} - \frac{p_\mu}{p \cdot k} \right] , \tag{5.12}$$

and the logarithmic structures

$$\Gamma^2_{(1)} = \frac{ie^2}{2} \int \frac{d^3k}{(2\pi)^3} \left[\frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right] \left[\frac{p'_\mu}{(p' \cdot k)^2} - \frac{p_\mu}{(p \cdot k)^2} \right] , . \tag{5.13}$$

$$\Gamma^3_{(1)} = \frac{ie^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \left[\frac{1}{p' \cdot k} + \frac{1}{p \cdot k} \right] . \tag{5.14}$$

Notice that the structure $\Gamma^2_{(1)}$ is the same as (4.30) on p. 95, and by Cauchy's theorem they are zero. The $\Gamma^1_{(1)}$ term has only linear IR structures which is the same as for 3+1 dimensions and, as we will see, can be cancelled using the BN arguments. We will see later when we calculate the total cross-section that the $\Gamma^3_{(1)}$ term does not cancel against real soft photons.

5.2 Bloch-Nordsieck in 3+1 Dimensions

In this section we shall first derive the expression that gives the amplitude for the emission of low energy photons in a process involving higher-energy massive particles at the lowest order in the coupling. We shall take a charged particle scattering off a potential. To be consistent, we will consider the case of scalar electrodynamics, although in 3+1 dimensions exactly the same results would be found in the fermionic theory. The relevant diagrams corresponding to this are shown in Figure 5.1. The diagrams 5.1(c) and 5.1(d) are irrelevant in the present discussion as they are IR finite. The diagrams corresponding to soft (and not separately observed) photons being emitted are shown in Figure 5.2.

5.2.1 Real Soft Photons

If we have an outgoing massive particle which emits a soft photon with outgoing momentum k, as in Figure 5.2(a), then we have a massive particle propagator carrying the momentum $p' + k$ before emitting the photon, and this makes the following contribution to the vertex:

$$\Gamma^{5.2a}_{(1)} = \mathcal{M}\frac{i(ie)(2p'+k)^\mu}{(p'+k)^2 - m^2 - i\varepsilon}, \tag{5.15}$$

where $\mathcal{M}$ is the amplitude for the tree level diagram. For on-shell external momenta, and in the limit $k \to 0$, we obtain the dominant term in 3+1 dimensions:

$$\Gamma^{5.2a}_{(1)} = -e\mathcal{M}\frac{p'^\mu}{p' \cdot k - i\varepsilon}. \tag{5.16}$$

If we have an incoming massive particle which emits a soft photon with outgoing momentum k, as shown in Figure 5.2(b), then we have a massive particle propagator carrying the momentum $p - k$, so in place of (5.16) we find a factor

$$\Gamma^{2b}_{(1)} = -e\mathcal{M}\frac{p^\mu}{-p \cdot k - i\varepsilon}, \tag{5.17}$$

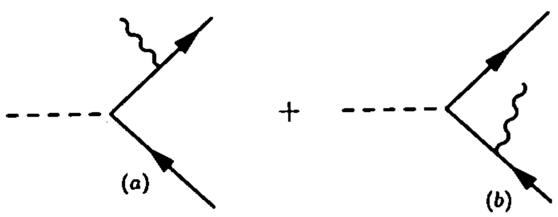


Figure 5.2: *Emission of real soft photon in the scattering of charges at one loop.*

where $\Gamma^{5.2a}_{(1)}$ and $\Gamma^{5.2b}_{(1)}$ represent the contributions of the diagrams 5.2(a) and 5.2(b) to the S-matrix. Combining (5.16) and (5.17) we obtain

$$\Gamma^{5.2}_{(1)} = -e\mathcal{M}\left\{\frac{p'^{\mu}}{p'\cdot k - i\varepsilon} - \frac{p^{\mu}}{p\cdot k + i\varepsilon}\right\}, \tag{5.18}$$

which is gauge invariant. Moreover, it is easy to show that this is true for charged particles of any spin. For example, for a particle of spin $\frac{1}{2}$ we have the following expression for the diagrams in Figure 5.2:

$$\Gamma^{5.2}_{(1)} = -e\left\{\gamma^{\mu}\frac{\not{p}' + \not{k} + m}{(p'+k)^2 - m^2 - i\varepsilon}\mathcal{M} + \mathcal{M}\frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2 - i\varepsilon}\gamma^{\mu}\right\}. \tag{5.19}$$

If we now use the identity $(\not{p}+m)\gamma^{\mu} = 2p^{\mu} - \gamma^{\mu}(\not{p}-m)$, take the external momenta on-shell, and the limit $k \to 0$, then we recover the expression (5.18).

We now find the cross-section for the emission of soft photons in the process. Since the cross-section is the square of the amplitude, we first calculate the amplitude, $\mathcal{M}_{\gamma}$, of this process, contracting the expression (5.18) with the polarisation vector. We obtain

$$\mathcal{M}_{\gamma} = -e\mathcal{M}\left\{\frac{p'^{\mu}}{p'\cdot k - i\varepsilon} - \frac{p^{\mu}}{p\cdot k + i\varepsilon}\right\}\epsilon_{\mu}, \tag{5.20}$$

where ϵ is the polarisation vector associated with the external photon line. The square of the amplitude is thus

$$|\mathcal{M}_\gamma|^2 = -e^2 |\mathcal{M}|^2 \left(\frac{p'^\mu}{p' \cdot k - i\varepsilon} - \frac{p^\mu}{p \cdot k + i\varepsilon} \right)^2 , \tag{5.21}$$

where we have used

$$\sum_{\text{polarisation}} \epsilon_\mu \epsilon_\nu^* = -g_{\mu\nu} \tag{5.22}$$

In order to use the BN approach, we need the cross-section for the emission of soft photons. To do this we simply substitute the resulting expression (5.21) for $|\mathcal{M}|^2$ into the standard cross-section formula (see for example Section 4.5 of [75]) and integrate over phase-space variables. We then obtain the cross-section for the emission of real soft photons, which is

$$\frac{d\sigma_\gamma}{d\Omega} = -e^2 \frac{d\sigma^{\text{Free}}}{d\Omega} \int_\lambda^E \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left(\frac{p'^\mu}{p' \cdot k - i\varepsilon} - \frac{p^\mu}{p \cdot k + i\varepsilon} \right)^2 , \tag{5.23}$$

where $d\sigma^{\text{Free}}/d\Omega$ represents the cross-section for the tree diagram and $\omega = |\underline{k}|$. We have also introduced the lower bound (or the IR cut-off) λ on the photon momenta while the upper bound E is the energy resolution of the experimental detector. Note that the emission of more than one soft photon would be of a higher order in the coupling.

5.2.2 Virtual Soft Photons

We now calculate the one loop cross-section for virtual soft photon emission. We use the technique outlined in Section 13.2 of [62], to factorise the virtual diagrams in terms of the real diagrams. In 2+1 dimensions we have sub-leading divergences and we will see later that a part of the sub-leading divergences cannot be factorised.

To calculate the amplitude for the virtual soft photon, we introduce a photon propagator factor

$$\frac{-ig_{\mu\nu}}{k^2 - i\varepsilon}, \tag{5.24}$$

then contract the gauge invariant expression in (5.18) by the same expression, with the product of the photon propagator, and integrate over the photon four momenta. For on-shell external momenta, and $k \to 0$, the dominant term arising from the one loop diagrams in Figure 5.1 is

$$\Gamma^{5.1}_{(1)} = \frac{e^2}{2}\mathcal{M}\int_{\lambda}^{\Lambda}\frac{d^4k}{(2\pi)^4}\left\{\frac{p'^{\mu}}{p'\cdot k - i\varepsilon} - \frac{p^{\mu}}{p\cdot k + i\varepsilon}\right\}\left\{\frac{p'^{\nu}}{-p'\cdot k - i\varepsilon} - \frac{p^{\nu}}{-p\cdot k + i\varepsilon}\right\}\frac{-ig_{\mu\nu}}{k^2 - i\varepsilon}. \tag{5.25}$$

To be consistent we have used the notation $\Gamma^{5.1}$ to represent the contribution of the diagrams in Figure 5.1. Here we have changed the sign of $p \cdot k$ and $p' \cdot k$ in the denominator of the second curly bracket, because k is reversed with respect to the first curly bracket. Also, we introduced a factor of $\frac{1}{2}$ because each of the diagrams 5.1(c) and 5.1(d) is reduced by a factor of $\frac{1}{2}$ due to $Z_2^{-1/2}$ factors. The upper cutoff, Λ, is arbitrarily chosen to distinguish between the IR and non-soft photons.

Now we link the diagrams to the S-matrix. The S-matrix is given by the expression (5.25) plus the tree level diagram. From this we calculate the cross-section to be

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\text{Free}}}{d\Omega}\left[1 + ie^2\int_\lambda^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)^2\right]. \tag{5.26}$$

5.2.3 Cancellation of IR Divergences

To see that the IR divergences cancel before doing any integration, we rewrite the cross-section for emitting the soft photons given by (5.23), by using the fact that:

$$\text{Re}\left(i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}f(|\underline{k}|,\underline{k})\right) = i\int \frac{d^3k}{(2\pi)^4}\frac{-2\pi i}{2|\underline{k}|}f(|\underline{k}|,\underline{k}). \tag{5.27}$$

Then

$$\frac{d\sigma_\gamma}{d\Omega} = -ie^2\frac{d\sigma^{\text{Free}}}{d\Omega}\int_\lambda^E \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\frac{p'^\mu}{p'\cdot k - i\varepsilon} - \frac{p^\mu}{p\cdot k + i\varepsilon}\right)^2. \tag{5.28}$$

Now we can combine these two cross-sections given by (5.25) and (5.28) and obtain the inclusive cross-section at order e^2:

$$\frac{d\sigma}{d\Omega} + \frac{d\sigma_\gamma}{d\Omega} = \frac{d\sigma^{\text{Free}}}{d\Omega}\left[1 + ie^2\int_E^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\frac{p'^\mu}{p'\cdot k - i\varepsilon} - \frac{p^\mu}{p\cdot k + i\varepsilon}\right)^2\right]. \tag{5.29}$$

Performing the integrals in the standard way, we obtain

$$\frac{d\sigma}{d\Omega} + \frac{d\sigma_\gamma}{d\Omega} = \frac{d\sigma^{\text{Free}}}{d\Omega}\left[1 + \text{A}\ln\left(\frac{E}{\Lambda}\right)\right], \tag{5.30}$$

where

$$\text{A} = \frac{\alpha}{\pi}\left(\frac{1}{v}\ln\left(\frac{1+v}{1-v}\right) - 2\right). \tag{5.31}$$

This does not depend on λ and we see that the IR divergences *cancel* at one loop. Since the IR divergences in 3+1 dimensions exponentiate, this can be shown to all orders in perturbation theory. We should also note that A is always positive. Since $E < \Lambda$, the resolution being a measure of what is soft, we see that the total cross-section becomes smaller ($\ln\left(\frac{E}{\Lambda}\right)$ is negative) as the resolution gets better. Thus in 3+1 dimensions, the BN approach yields an IR finite total, inclusive cross-section. This approach is radical because it does not attempt to construct an S-matrix and defines QED purely at the level of such cross-sections.

5.3 Bloch-Nordsieck in 2+1 Dimensions

In this section we will study the Bloch-Nordsieck approach in 2+1 dimensions. We will have to consider separately the linear and the logarithmic IR divergences. Again we consider the case of scalar electrodynamics.

5.3.1 Real Soft Photons

We will first consider real soft photon emission as in Figure 5.2. Simple power counting shows that we shall need to take account of more terms.

As in the case of 3+1 dimensions, we need to find the gauge invariant structures for real soft photon emission in scattering off a current. For on shell external momenta the diagrams in Figure 5.2 give the following contribution:

$$\Gamma^{5.2}_{(1)} = -e\mathcal{M}\left[\frac{(2p'+k)^{\mu}}{k^2+2p'\cdot k}+\frac{(2p-k)^{\mu}}{k^2-2p\cdot k}\right]. \tag{5.32}$$

Expanding around soft k, we find the following dominant terms in 2+1 dimensions:

$$\Gamma^{5.2}_{(1)} = -e\mathcal{M}\left\{\left[\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right] + \frac{1}{2}\left[\frac{k^{\mu}}{p'\cdot k} - \frac{p'^{\mu}}{p'\cdot k}\frac{k^2}{p'\cdot k} + \frac{k^{\mu}}{p\cdot k} - \frac{p^{\mu}}{p\cdot k}\frac{k^2}{p\cdot k}\right]\right\}. \tag{5.33}$$

It is clear that both the structures in this expression are separately gauge invariant. The first square bracket is the same as in 3+1 dimensions and is therefore spin independent. The new sub-leading contribution however is no longer spin independent. To make this explicit, in fermionic QED we now find a different gauge invariant structure:

$$\begin{aligned}\Gamma^{5.2}_{(1)} &= -e\mathcal{M}\left\{\left[\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right]\right.\\ &+ \left.\frac{1}{2}\left[\frac{\gamma^{\mu}\not{k}}{p'\cdot k}\mathcal{M} - \mathcal{M}\frac{p'^{\mu}}{p'\cdot k}\frac{k^2}{p'\cdot k} + \mathcal{M}\frac{\not{k}\gamma^{\mu}}{p\cdot k} - \mathcal{M}\frac{p^{\mu}}{p\cdot k}\frac{k^2}{p\cdot k}\right]\right\}.\end{aligned} \tag{5.34}$$

As before we want to calculate the cross-section for the emission of soft photons and we continue with scalar QED.

We calculate the amplitude $\mathcal{M}_{\gamma}$, which is, by definition, the contraction of the gauge invariant factor (5.33) with the polarisation vector. We obtain

$$\Gamma^{5.2}_{(1)} = -e\mathcal{M}\left\{\left[\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right] + \frac{1}{2}\left[\frac{k^{\mu}}{p'\cdot k} - \frac{p'^{\mu}}{p'\cdot k}\frac{k^2}{p'\cdot k} + \frac{k^{\mu}}{p\cdot k} - \frac{p^{\mu}}{p\cdot k}\frac{k^2}{p\cdot k}\right]\right\}\epsilon_{\mu}\,. \tag{5.35}$$

To calculate the cross-section, we square the amplitude and obtain:

$$\begin{aligned}\frac{d\sigma_{\gamma}}{d\Omega} &= -e^2\frac{d\sigma^{\mathrm{Free}}}{d\Omega}\int_{\lambda}^{E}\frac{d^2k}{(2\pi)^2}\frac{1}{2\omega}\left\{\left[\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right]^2\right.\\ &\quad\left. - k^2\left[\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right]\left[\frac{p'_{\mu}}{(p'\cdot k)^2} + \frac{p_{\mu}}{(p\cdot k)^2}\right]\right\},\end{aligned} \tag{5.36}$$

where we have dropped the square of the sub-leading terms which are finite. The minus sign in front is again a consequence of (5.22). We have dropped the k^μ terms from the sub-leading part of (5.35) because they vanish by gauge invariance. The upper bound, E, is the energy resolution of the detector. The lower bound, λ, is the small IR cut-off. The second term in the above expression is new while the first one is equivalent to (5.25), though now it is linearly IR divergent.

5.3.2 Virtual Soft Photons

The sub-leading terms in (5.35) cannot be factorised, so the technique used in [62] to calculate the virtual photons exchanged will not work for the sub-leading divergences in 2+1 dimensions.

We now calculate the contribution to the cross-section from virtual soft photons using the usual technique, i.e., calculate the virtual diagrams using the Feynman rules and then extract all the IR divergences. From (5.11) on page 107, the IR divergence expression for virtual diagrams in 2+1 dimensions.

$$\Gamma_{(1)} = \Gamma^1_{(1)} + \Gamma^3_{(1)}\,, \tag{5.37}$$

where

$$\Gamma^1_{(1)} = \frac{ie^2}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left[\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right]\left[\frac{p'_\mu}{p'\cdot k} - \frac{p_\mu}{p\cdot k}\right]\,, \tag{5.38}$$

and

$$\Gamma^3_{(1)} = \frac{ie^2}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left[\frac{1}{p'\cdot k} + \frac{1}{p\cdot k}\right]\,. \tag{5.39}$$

The $\Gamma^2_{(1)}$ term in (5.11) can be dropped, since it vanishes by Cauchy's theorem. To calculate the cross-section we now link the diagrams to the S-matrix, which we calculated in (5.37), plus the tree level one, and so obtain

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\text{Free}}}{d\Omega}\left\{1 + ie^2\int_\lambda^\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left[\left(\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right)^2 + \left(\frac{1}{p'\cdot k} + \frac{1}{p\cdot k}\right)\right]\right\}. \quad (5.40)$$

5.3.3 The Inclusive Cross-Section

In order to try to cancel the IR divergences we add the two cross-sections (5.36) and (5.40). To do this first convert (5.36) from two to three dimensions with the help of (5.27), and then drop the sub-leading term. We obtain

$$\frac{d\sigma_\gamma}{d\Omega} = -ie^2\frac{d\sigma^{\text{Free}}}{d\Omega}\int_\lambda^E \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left[\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right]^2. \quad (5.41)$$

Now we can combine (5.40) and (5.41) to calculate the total cross-section for a system of particles scattering off a potential in 2+1 dimensions and so obtain,

$$\begin{aligned}\frac{d\sigma}{d\Omega} + \frac{d\sigma_\gamma}{d\Omega} = \frac{d\sigma^{\text{Free}}}{d\Omega}&\left\{1 + ie^2\int_E^\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left(\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right)^2\right.\\ &\left. + ie^2\int_\lambda^\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left(\frac{1}{p'\cdot k} + \frac{1}{p\cdot k}\right)\right\}. \end{aligned} \quad (5.42)$$

The linear term does not depend on λ and so the leading IR divergences cancel at one loop in 2+1 dimensions but the sub-leading, logarithmic IR divergence, i.e., the second term in the above expression, does *not* vanish. We are forced to conclude that the Bloch-Nordsieck method does not solve the IR problem in 2+1 dimensions.

Let us explain diagrammatically why the sub-leading divergences do not cancel in 2+1 dimensions. When we calculate the cross-section for real soft photon emission we square the sum of the two diagrams in Figure 5.2 and obtain the sets of diagrams in Figure 5.3.

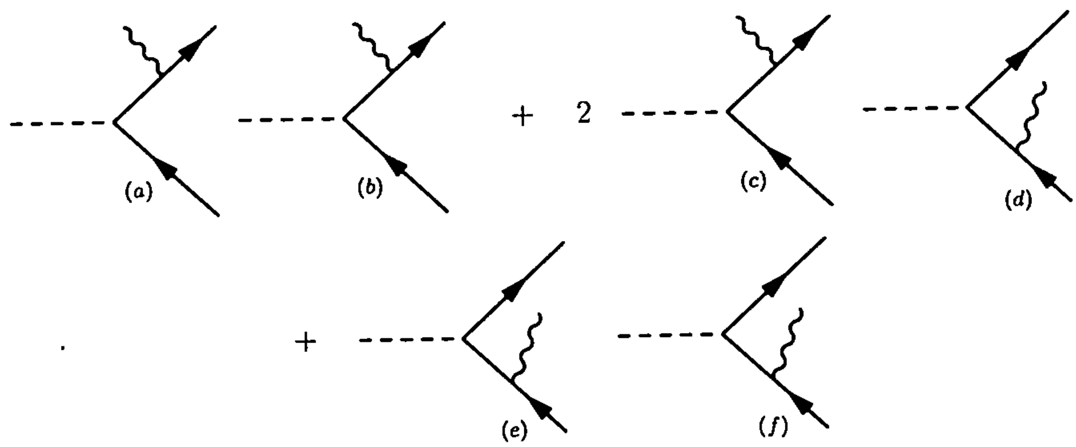


Figure 5.3: *Square of the emission of real soft photon in the scattering of charges at one loop.*

When we factorise the virtual diagram 5.1(a) we obtain the product of the real diagrams 5.3(c) and 5.3(d). This cancels both the leading and sub-leading IR divergences exactly. However the virtual diagram 5.1(c) factorises as the product of 5.3(a) and 5.3(b) only to the level of leading IR divergences, and similarly for virtual diagram 5.1(d) as the product of 5.3(e) and 5.3(f). Weinberg, (p. 538 of [62]) argued that the self-energy diagram can also be found by factorising two photons, emitting from the same external line as for example, when a photon with momentum k_1 is emitted from an external line of momentum p after a photon with momentum k_2. In

3+1 dimensions we obtain the factor

$$\frac{\eta e p^{\mu}}{p \cdot k_1} \frac{\eta e p^{\nu}}{p \cdot (k_2 + k_1)} \,. \tag{5.43}$$

If k_2 is emitted after k_1 then the factor is

$$\frac{\eta e p^{\mu}}{p \cdot k_2} \frac{\eta e p^{\nu}}{p \cdot (k_1 + k_2)} \,, \tag{5.44}$$

where η is a sign factor with the value +1 for outgoing particle and -1 for incoming particle. Adding (5.43) and (5.44) we obtain

$$\frac{\eta e p^{\mu}}{p \cdot k_1} \frac{\eta e p^{\nu}}{p \cdot k_2} \,, \tag{5.45}$$

which is a factor for the self-energy diagram when $k_1 = k_2$.

In 2+1 dimensional case this works only up to the level of leading IR divergences. In 2+1 dimensions therefore, there are sub-leading IR divergences which we would not expect to be cancelled by the BN method. This explains why they did not cancel in the explicit calculation (5.42).

Here we are considering scattering off a current rather than a photon. In particular, we have the logarithmic IR divergent term (5.39), which does not cancel even in the limit of zero momentum transfer, since when $p = p'$, this becomes,

$$\Gamma_3(p = p') = ie^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \frac{1}{p \cdot k} \,. \tag{5.46}$$

Thus there is a sub-leading IR divergent term surviving in the zero momentum transfer limit. This is rather unusual and does not occur in 3+1 dimensions.

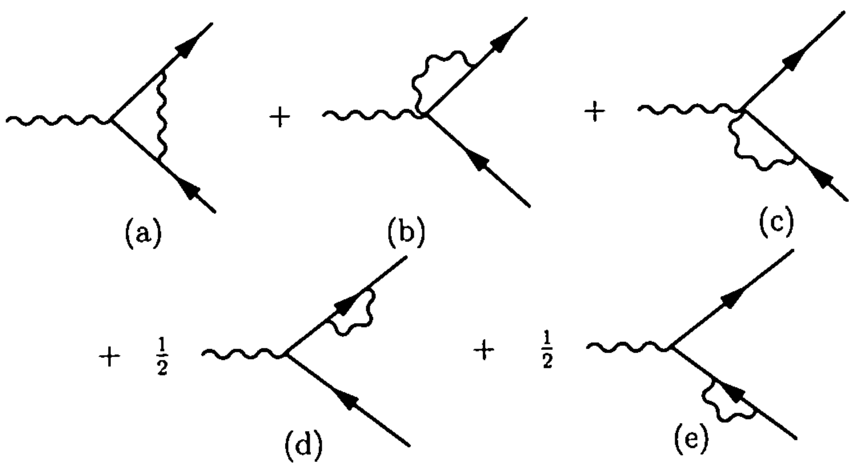


Figure 5.4: *The relevant vertex correction diagrams in scalar QED at the level of the S-matrix.*

To investigate this, we now study the photon scattering vertex, i.e., a massive particle scattering off a photon which corresponds more closely to physical Coulomb scattering. The diagrams involved are now given by Figure 5.4 where the four-point interaction of scalar QED has introduced new diagrams.

Before proceeding, we check the gauge invariance of this S-matrix element. As in Section 5.1, we first calculate the diagrams of Figure 5.4 to obtain the desired gauge invariant structure. After a little bit of algebra, the gauge dependent part, in the Lorentz class of gauge, is proportional to

$$-\xi e^3 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^4} \left[\frac{1}{2m^2} (p' \cdot k + p \cdot k)(p + p')^\mu - k^\mu \right] . \tag{5.47}$$

This is an *odd* massless tadpole which vanishes upon integration, confirming the gauge invariance of the S-matrix for matter scattering off a photon.

Next, we extract the IR divergences associated with the diagrams in Figure 5.4. We obtain

$$\Gamma(p,p') = \frac{ie^2}{2}(p+p')^\mu \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}\left\{\left[\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right]^2 + \left[\frac{p\cdot k + p'\cdot k}{p\cdot k\, p'\cdot k}\,\frac{m^2 - p\cdot p'}{m^2 + p\cdot p'}\right]\right\}. \tag{5.48}$$

The first square bracket in the above expression has only linear IR structures, and the second square bracket has only logarithmic structures.

To be consistent, we first calculate the cross-section for real soft photon emission. For the process of matter scattering off a photon the diagrams corresponding to real soft photons being emitted are the same as before, i.e., the diagrams in Figure 5.2, plus a four-point vertex. For on-shell external momenta, the contribution of these diagrams to the vertex is

$$\Gamma^{\text{real}}_{(1)} = -ie^2\mathcal{M}\left[\frac{(2p'+k)^\rho \epsilon^k_\rho (p+p'+k)^\mu}{k^2+2p'\cdot k} + \frac{(2p'-k)^\rho \epsilon^k_\rho (p+p'-k)^\mu}{k^2-2p\cdot k} - 2\epsilon^\mu_k\right]\epsilon^q_\mu\,. \tag{5.49}$$

In order to show that the result is gauge invariant, we take the longitudinal part of the polarisation vector, i.e., ϵ^μ_k is replaced with k^μ in the above expression (5.49) (a discussion of this can be found in Section 5.5 of [75]), and obtain

$$-ie^2\mathcal{M}[(p+p'+k)^\mu - (p+p'-k)^\mu - 2k^\mu] = 0\,. \tag{5.50}$$

This makes manifest the gauge invariance of the real soft emission for the process of matter scattering off a photon.

We now extract the IR divergences from (5.49), obtaining the amplitude

$$-ie^2\mathcal{M}\left[\left(\frac{p'^\rho}{p'\cdot k}-\frac{p^\rho}{p\cdot k}\right)(p+p')^\mu+\left(\frac{p'^\rho}{p'\cdot k}+\frac{p^\rho}{p\cdot k}\right)k^\mu \right.$$
$$\left.+\frac{k^\rho}{2}\left(\frac{1}{p'\cdot k}+\frac{1}{p\cdot k}\right)-2g^{\mu\rho}\right]\epsilon_\rho\,. \tag{5.51}$$

To calculate the cross-section, we square the amplitude (5.51) for Coulomb scattering (see also page 94 of [73]), setting the spatial components of the potential to zero, i.e. $\mu = 0$. We obtain

$$\frac{d\sigma_\gamma}{d\Omega}=e^4\frac{d\sigma^{\text{Free}}}{d\Omega}\int_\lambda^E\frac{d^2k}{(2\pi)^2}\frac{(p^0+p'^0)}{2\omega}\left\{\left[\frac{p'}{p'\cdot k}-\frac{p}{p\cdot k}\right]^2(p^0+p'^0)\right.$$
$$\left.-\,2k^0\left[\frac{p'^2}{p'\cdot k}-\frac{p^2}{p\cdot k}\right]-4\left[\frac{p'^0}{p\cdot k}-\frac{p^0}{p\cdot k}\right]\right\}, \tag{5.52}$$

where we have used the fact that $\epsilon_\mu\epsilon_\nu^* = -g_{\mu\nu}$. The first term in the above expression has linear IR infinities, while the second and third terms have logarithmic IR infinities.

There is a simple argument to show that the second term in (5.52) vanishes. The integrals

$$\int\frac{d^2k}{p\cdot k}\quad\text{and}\quad\int\frac{d^2k}{p'\cdot k} \tag{5.53}$$

are equivalent to each other since these are scalar integrals, and the only possible scalar quantity is either p^2 or p'^2. For on-shell external momenta $p^2 = p'^2 = m^2$ and the integrals cancel.

If we use the same argument, we see that the third integral is proportional to

$$(p^{0^2}-p'^{0^2})\int\frac{d^3k}{k^2}\frac{1}{p\cdot k}\,. \tag{5.54}$$

In the Breit frame, (i.e., $p = -p'$), this is equal to zero, so the cross-section for the emission of a soft photon is

$$\frac{d\sigma_\gamma}{d\Omega} = ie^4 \frac{d\sigma^{\text{Free}}}{d\Omega} \int_\lambda^E \frac{d^3k}{(2\pi)^3} \frac{(p^0 + p'^0)^2}{k^2} \left[\frac{p'}{p' \cdot k} - \frac{p}{p \cdot k} \right]^2 , \tag{5.55}$$

where we have used (5.27) to convert it into a three dimensional integral.

Next we calculate the cross-section for the virtual soft photon emission. To do this we take the square of the sum of the diagrams in Figure 5.4 and the tree level one, and we obtain

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma^{\text{Free}}}{d\Omega} \left\{ 1 - \int_\lambda^E \frac{d^3k}{(2\pi)^3} \frac{(p^0 + p'^0)^2}{k^2} \left[\frac{p'}{p' \cdot k} - \frac{p}{p \cdot k} \right]^2 \right.$$
$$\left. - \int_\lambda^E \frac{d^3k}{(2\pi)^3} \frac{(p^0 + p'^0)^2}{k^2} \left[\frac{p \cdot k + p' \cdot k}{p \cdot k p' \cdot k} \frac{m^2 - p \cdot p'}{m^2 + p \cdot p'} \right] \right\} . \tag{5.56}$$

We now see that at zero momentum transfer both (5.55) and (5.56) vanish. This shows that our results preserve the Ward Identity. This also shows that in 2+1 dimensions the detailed structure of the vertex changes the IR properties of the S-matrix. However, if we combine (5.55) and (5.56) we see that the leading IR divergences cancel but the sub-leading term does not cancel. This shows that even in the case of Coulomb scattering the Bloch-Nordsieck method does not yield an IR finite total cross-section.

5.4 Summary

In this chapter we have applied the BN method to the IR divergences of the scattering of matter in 3+1 and 2+1 dimensions in scalar QED. We have shown that the Bloch-Nordsieck approach breaks down in 2+1 dimensions. This method is also known to break down in 3+1 dimensions for collinear divergences, i.e., with massless charges and is a major source of difficulties in QCD calculations [76]. A discussion of this can also be found in the Section 13.4 of [62]. In general, these collinear divergences are not eliminated by summing over real soft emissions. However Kinoshita, Lee and Nauenberg [77,78] argued that these divergences can be made to cancel if we sum over real soft photon emission in the final states and incoming photons in the appropriate initial states. The question remains as to whether there is a way of getting an IR finite inclusive cross-sections in 2+1 dimensions by adding appropriate initial states.

Chapter 6

Conclusion

6.1 Discussion

In this thesis we have studied the IR problems that appear in 2+1 dimensional gauge theories. Let us briefly recall some of the important results that we have observed.

Our starting point was QED in 3+1 dimensions. One loop calculations of various Green's function were presented and we discussed how the IR divergences occur in 3+1 dimensions. We then reviewed an array of responses to the IR problem. We sketched the Bloch-Nordsieck method and then considered the formalism of Kulish and Faddeev [10] to discuss the physical origin of the IR problem. They argued that switching off the coupling at large distances is not valid. This, we argued, implied that the matter fields in the original Lagrangian cannot be identified with the asymptotic physical fields. This led us to the framework introduced by the Plymouth group, which was based on describing the charged particles in relativistic QED through a process of dressing the matter with the appropriate electromagnetic field. To obtain the full structure of the dressing, we solved the dressing equation (2.53) while demanding that the dressed matter be gauge invariant (2.47). We showed that there are two factors in the dressing, a gauge dependent part, which was necessary for gauge invariance and a gauge independent part, which was essential to fulfill the dressing equation. One loop calculation of the on-shell Green's functions with the dressed field was repeated and we saw that they were free of IR divergences.

We began our studies of gauge theories in 2+1 dimensions in Chapter 3. We analysed the IR properties of the various on-shell Green's functions. To study the spin dependence of the renormalisation constants, both spinor and scalar theories were considered. We saw that, as expected from power counting, there were serious IR problems in 2+1 dimensions. The mass shift now picks up divergences in the IR region. In addition, the IR divergences in the wave function renormalisation constant are worse: there are linear, as well as logarithmic, IR divergences in the fermionic theory. We employed different regularisation schemes to regulate the IR divergences and found that the leading IR divergences were spin independent, while the sub-leading divergences were spin dependent. We also calculated the renormalisation constants associated with the vertex correction diagram and verified that they preserved the Ward identity. To understand the gauge dependence, we studied the Green's functions in different gauges.

In Chapter 4, we used the dressing method to study the physical fields of QED in 2+1 dimensions. We observed that if we use the full dressing to solve the dressing equation, then the mass shift and the wave function renormalisation constant were IR finite. These results were calculated in both fermionic and scalar QED. This shows that it is possible to construct a gauge independent and IR finite description of charge propagation in 2+1 dimensions. This may be useful in condensed matter physics [32–34]

We calculated the IR properties of the scattering of dressed charges in 2+1 dimensional scalar QED and showed that the IR structures were gauge invariant. They all cancelled for zero momentum transfer, which was as expected from the Ward Identity. There were, however, IR linear divergences associated with even massless tadpoles that do not cancel for non-zero momentum transfer. These massless tadpoles also exist in 3+1 dimensions, where they vanish due to dimensional regularisation.

Finally, in Chapter 5, we used the Bloch-Nordsieck method to study the cancellation of IR divergences at the level of inclusive cross-section in both 3+1 and 2+1 dimensional QED. This is the most common approach in 3+1 dimensions. However, we showed that the BN approach breaks down in 2+1 dimensions. (It is also known to break down in the presence of collinear divergences in 3+1 dimensions.) The form of the sub-leading divergences in 2+1 dimensions were seen to depend on the type of vertex that we utilise (in 3+1 dimensions it was clear that the IR divergences were independent of the vertex).

6.2 Future Work

This thesis has analysed the IR divergences that arise in 2+1 dimensional QED. There remains many unanswered questions, some of the most important of them are as follows.

First, the proof of the IR finiteness of the dressed on-shell Green's functions should be extended to all orders in perturbation theory. In 3+1 dimensions this is relatively easy due to the exponentiation of the IR divergences. In 2+1 dimensions it is not clear that such an exponentiation takes place, although explicit calculations [64] indicate that the IR problem at higher orders is not as severe as power counting implies. Sen [64] calculated the mass shift at two loops and showed that the leading IR infinities cancel with those of wave function renormalisation constants at one loop. It is not yet clear why these cancellations take place. Two loop calculations for the wave function renormalisation still need to be performed. Work beyond two loops would also be important.

The full renormalisation of the 2-point Green's function with the full dressing should be performed. This could be useful in condensed matter studies.

We have seen in Chapter 3 that dimensional regularisation regulates logarithmic divergences and sets all power divergences to zero. An important and still open problem is how to modify the dressing for a regularisation scheme other than dimensional regularisation. This would be interesting in both 3+1 and 2+1 dimensions.

The Kulish and Faddeev calculation of the asymptotic Hamiltonian must change in 2+1 dimensions, since there the IR structures are very different and become spin dependent. Naively repeating their argument in 2+1 dimensions does not seem to alter their conclusions, i.e., that $\mathcal{H}^{\rm as}_{\rm int}$ is spin independent and this seems to contradict perturbative calculations. Clearly a more sophisticated study is needed. The IR structure in 2+1 dimensions is much richer than in 3+1 dimensions and this may help to explain the asymptotic Hamiltonian of QCD [76]. This is a key question because in 2+1 dimensions, the force between charges varies as $1/r$, so the amount of work needed to separate them grows logarithmically with distance [31].

More work is needed on the analysis of the breakdown of the Bloch-Nordsieck method in 2+1 dimensions. In particular, it would be useful to find a way of obtaining IR finite inclusive cross-sections by adding appropriate initial states, like the KLN arguments on cancellation of collinear divergences.

Finally, the demonstration that the screening interaction in 3+1 dimensional QED [19] is due to the additional dressing structure (K) should be repeated in 2+1 dimensions.

Appendix A

Gauge Invariance of the Dressed Propagator

The purpose of this appendix is to show the gauge invariance of the dressed propagator in a rather simple and attractive way. We first consider the one loop Feynman diagrams, which are shown in Figure A.1. Note that we only include here the minimal dressing, since the additional dressing is itself gauge invariant.

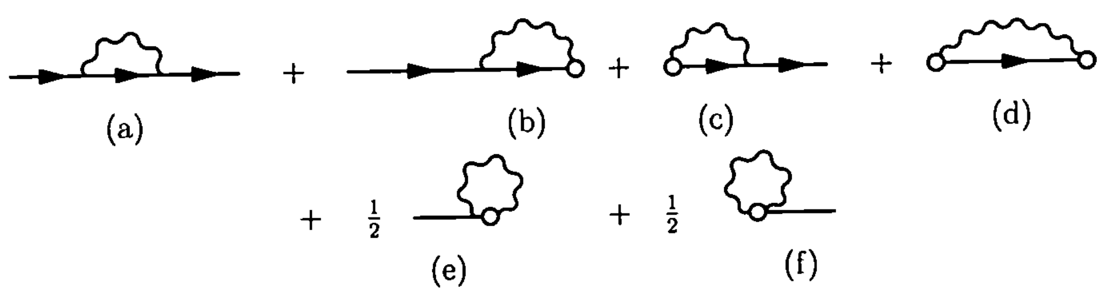


Figure A.1: *All the one-loop Feynman diagrams in the electron propagator when we include the minimal dressing.*

The first step is to write down the contribution of each diagram. To do this we recall the Feynman rules described in Section 2.1 and the rule for the dressed vertex, which can be found in Figure 4.1 in Section 4.1.

The contribution of the covariant diagram to the propagator is

$$-i\Sigma^a(p) = -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma_\nu}{(p-k)^2 - m^2} D^{\mu\nu}(k)\,. \tag{A.1}$$

Here the photon propagator is left completely general and we are evaluating the integrals in D dimensions. The diagrams A.1(b) and A.1(c) contribute respectively

$$-i\Sigma^b(p) = e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma_\mu(\not{p} - \not{k} + m)(\not{p} - m)V_\nu}{[(p-k)^2 - m^2]V\cdot k} D^{\mu\nu}(k)\,, \tag{A.2}$$

and

$$-i\Sigma^c(p) = e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{(\not{p} - m)(\not{p} - \not{k} + m)\gamma_\mu V_\nu}{[(p-k)^2 - m^2]V\cdot k} D^{\mu\nu}(k)\,. \tag{A.3}$$

The rainbow diagram, A.1(d), generates

$$-i\Sigma^d(p) = -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{(\not{p} - m)(\not{p} - \not{k} + m)(\not{p} - m)V_\mu V_\nu}{[(p-k)^2 - m^2](V\cdot k)^2} D^{\mu\nu}(k)\,, \tag{A.4}$$

and finally the contribution of the two tadpole diagrams, A.1(e) and A.1(f), is together

$$-i\Sigma^{e+f}(p) = e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{(\not{p} - m)V_\mu V_\nu}{(V\cdot k)^2} D^{\mu\nu}(k)\,. \tag{A.5}$$

Consider the rainbow diagram (A.4). This is ill-defined as a result of off-shell IR divergences in both 3+1 and 2+1 dimensions. We thus rewrite it with the help of the algebraic identity:

$$\frac{1}{\not{p} - \not{k} - m} = \frac{1}{\not{p} - m}\left[1 + \not{k}\frac{1}{\not{p} - \not{k} - m}\right]\,, \tag{A.6}$$

and so obtain

$$-i\Sigma^d(p) = -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{V_\mu V_\nu D^{\mu\nu}(k)}{(V\cdot k)^2} \left[1 + \frac{\not{k}(\not{p}-\not{k}+m)(\not{p}-m)}{[(p-k)^2-m^2]}\right] . \tag{A.7}$$

The term proportional to one in (A.7) will be cancelled by tadpoles (A.5) and so we write the rainbow diagram as

$$-i\Sigma^d(p) = -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{V_\mu V_\nu D^{\mu\nu}(k)}{(V\cdot k)^2} \frac{\not{k}(\not{p}-\not{k}+m)(\not{p}-m)}{[(p-k)^2-m^2]} . \tag{A.8}$$

This is now IR finite off-shell in 3+1 dimensions and we stop here. In 2+1 dimensions the procedure needs repeated here and in (A.2) but our final result for the sum of all the diagrams will be the same.

We now simplify the numerator in (A.2), (A.3) and (A.8). The easiest way to do this is by adding and subtracting $\not{k}$ from the $(\not{p}-m)$ term. This then gives

$$(\not{p}-\not{k}+m)(\not{p}-\not{k}-m) + (\not{p}-\not{k}+m)\not{k} = (p-k)^2 - m^2 + (\not{p}-\not{k}+m)\not{k} . \tag{A.9}$$

The first structure in (A.9), is an *odd* massless tadpole and therefore can be dropped. By putting together all the diagrams in Figure A.1:

$$-i\Sigma(p) = -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma_\nu}{(p-k)^2-m^2} \left(g_{\rho\mu} - \frac{V_\rho k_\mu}{V\cdot k}\right) D^{\rho\sigma}(k) \left(g_{\sigma\nu} - \frac{V_\sigma k_\nu}{V\cdot k}\right) . \tag{A.10}$$

This brings out the gauge invariance of the dressed propagator. We can then replace $D^{\rho\sigma}(k)$ by $g^{\rho\sigma}/k^2$ and obtain the final result

$$-i\Sigma(p) = -e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma_\nu}{[(p-k)^2-m^2]k^2} \left(g^{\mu\nu} - \frac{V^\mu k^\nu + k^\mu V^\nu}{V\cdot k} + \frac{V^2 k^\mu k^\nu}{(V\cdot k)^2}\right) , \tag{A.11}$$

which is exactly the self-energy with the dressing gauge propagator.

Using similar arguments in scalar QED, we rapidly find that the self-energy of the dressed field has the following gauge invariant form

$$-i\Sigma(p) = -e^2 \int \frac{d^D k}{(2\pi)^D} \frac{(2p-k)^\mu (2p-k)^\nu}{(p-k)^2 - m^2} \left(g_{\rho\mu} - \frac{V_\rho k_\mu}{V \cdot k} \right) D^{\rho\sigma}(k) \left(g_{\sigma\nu} - \frac{V_\sigma k_\nu}{V \cdot k} \right) . \tag{A.12}$$

Appendix B

Pauli-Villars Regularisation for UV Divergences in 2+1 Dimensions.

Although the main aim of this thesis is to study IR structures in 2+1 dimensions, in this appendix we employ the Pauli-Villars regulator to study the UV divergences associated with the scalar electron propagator in 2+1 dimensions. In particular, we will calculate the mass shift, which is known to have UV divergences in scalar QED. We also want to check that there are no UV divergences associated with the wave function renomalisation constant. We note that UV divergences in fermionic QED will be absent at one loop [40].

In order to use the Pauli-Villars regulator we first modify the photon propagator. We redefine the Lagrangian density in an arbitrary Lorentz gauge as follows (see also Section 17.2 of [60]):

$$\mathcal{L}_{\rm reg} = -\frac{1}{4}F^{\mu\nu}\left(\frac{\Box + M^2}{M^2}\right)F_{\mu\nu} - \frac{1}{2\xi}\partial_\mu A^\mu\left(\frac{\Box + M^2}{M^2}\right)\partial_\nu A^\nu\,. \tag{B.1}$$

This regularisation is gauge invariant because the field strength $F^{\mu\nu}$ is gauge invariant in QED. As $M^2 \to \infty$, we regain the standard theory. This Lagrangian can now, for our purposes, be coupled to matter in the usual way. It is important to note that the Pauli-Villars regularisation will not maintain gauge invariance in QCD, because the field strength is not gauge invariant in a non-abelian theory.

To determine the modified photon propagator we look at the part of the Lagrangian that is quadratic in the photon field. In momentum space it may be written as

$$\mathcal{L}_{reg} = \frac{1}{2M^2} A^\mu \left[g_{\mu\nu} k^2 (k^2 - M^2) + (1 - \frac{1}{\xi})^2 k^2 k_\mu k_\nu - 2(1 - \frac{1}{\xi}) k^2 k_\mu k_\nu + \mu^2 (1 - \frac{1}{\xi}) k_\mu k_\nu \right] A^\nu \tag{B.2}$$

From this we obtain, after a little algebra,

$$D^{reg}_{\mu\nu} = \frac{1}{k^2 - M^2} \left[g_{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \frac{k_\mu k_\nu}{k^2 - \xi M^2} \right] - \frac{1}{k^2} \left[g_{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \frac{k_\mu k_\nu}{k^2} \right] . \tag{B.3}$$

The fact that the propagator can be written on this difference underlies the success of the Pauli-Villars scheme.

The UV divergences

As we mentioned earlier, the one loop matter propagator in scalar QED diverges in the UV region. To carry out this perturbative calculation, we now calculate the UV part of the mass shift and the wave function renormalisation constant in Feynman gauge. For this we need to consider Figures 3.4(a) and 3.4(b). Figure 3.4(b) only has UV divergences while Figure 3.4(a) has both UV and IR divergences.

As well as the Pauli-Villars regulator, we introduce a small photon mass μ as an IR cut-off. This will enable us to check our previous results in Chapter 3. The photon propagator, (B.3) in Feynman gauge then becomes

$$-iD_{\mu\nu} = -ig_{\mu\nu}\left(\frac{1}{k^2-\mu^2} - \frac{1}{k^2-M^2}\right). \tag{B.4}$$

We take the limit $M \to \infty$ at the end of the calculation to extract the UV divergences.

The expression for diagram 3.4(a) is

$$-i\Sigma^{3.4a}(p) = e^2 \int \frac{d^3k}{(2\pi)^3}\left(\frac{i}{k^2-\mu^2} - \frac{i}{k^2-M^2}\right)\frac{i(2p-k)^2}{(p-k)^2-m^2}. \tag{B.5}$$

Let us now consider the first term of the photon propagator. (To calculate the second term, we merely substitute M^2 for μ^2).

To use the Pauli-Villar regulator, we found it simplest to use the Schwinger trick (see also page 387 of [60]) rather than the Feynman trick (see for example page 157 of [79]). The idea is to evaluate the integrals by exponentiating the denominators. We write

$$\frac{i}{k^2-\mu^2} = \int_0^\infty dx\, e^{ix(k^2-\mu^2)}, \tag{B.6}$$

and

$$\frac{i}{(p-k)^2-m^2} = \int_0^\infty dy\, e^{iy((p-k)^2-m^2)}. \tag{B.7}$$

Using these in the first term of (B.5), we obtain

$$-i\Sigma^{3.4a}_\mu(p) = e^2 \int_0^\infty dx \int_0^\infty dy \int \frac{d^3k}{(2\pi)^3}(2p-k)^2 e^{ix(k^2-\mu^2)} e^{iy((p-k)^2-m^2)}. \tag{B.8}$$

Next, we complete the square in the exponentials with respect to k and obtain, after shifting the integration variable $k \to k + py/(x+y)$,

$$-i\Sigma_{\mu}^{3.4a}(p) = e^2 \int_0^{\infty} dx \int_0^{\infty} dy \int \frac{d^3k}{(2\pi)^3} \left[\left(2 - \frac{y}{x+y}\right) p - k\right]^2 e^{ik^2(x+y)} \times \exp\left[i\left(\left(\frac{p^2xy}{x+y}\right) - m^2y - \mu^2x\right)\right] . \quad \text{(B.9)}$$

The k integral is a standard Gaussian integral and therefore easy to perform. After dropping the odd integral, we obtain

$$-i\Sigma_{\mu}^{3.4a}(p) = \frac{e^2}{8\pi^2}\sqrt{\frac{\pi}{i}} \int_0^{\infty} dx \int_0^{\infty} dy \left[\frac{\left(2 - \frac{y}{x+y}\right)^2 p^2}{(x+y)^{3/2}} + \frac{3i}{2}\frac{1}{(x+y)^{5/2}}\right] \times \exp\left[i\left(\left(\frac{p^2xy}{x+y}\right) - m^2y - \mu^2x\right)\right] . \quad \text{(B.10)}$$

We now introduce the algebraic identity

$$1 = \int_0^{\infty} \frac{d\beta}{\beta} \delta\left(1 - \frac{x+y}{\beta}\right) , \quad \text{(B.11)}$$

and then change $x \to \beta x$ and $y \to \beta y$ to rewrite

$$-i\Sigma_{\mu}^{3.4a}(p) = \frac{e^2}{8\pi^2}\sqrt{\frac{\pi}{i}} \int_0^{\infty} dx \int_0^{\infty} dy \int_0^{\infty} \frac{d\beta}{\beta^{1/2}} \left[\frac{\left(2 - \frac{y}{x+y}\right)^2 p^2}{(x+y)^{3/2}} + \frac{3i}{2\beta}\frac{1}{(x+y)^{5/2}}\right] \times \delta(1-x-y) \exp\left[i\beta\left(\left(\frac{p^2xy}{x+y}\right) - m^2y - \mu^2x\right)\right] \quad \text{(B.12)}$$

We now perform one of the parametric integrations by using the delta function to obtain

$$-i\Sigma_{\mu}^{3.4a}(p) = \frac{e^2}{8\pi^2}\sqrt{\frac{\pi}{i}} \int_0^{1} dx \int_0^{\infty} \frac{d\beta}{\beta^{1/2}} \left[(1+x)^2p^2 + \frac{3i}{2\beta}\right] e^{i\beta a} , \quad \text{(B.13)}$$

where $a = p^2x(1-x) - m^2(1-x) - \mu^2(1-x)$.

We repeat these steps to calculate the second part of the photon propagator and obtain

$$-i\Sigma_M^{3.4a}(p) = \frac{e^2}{8\pi^2}\sqrt{\frac{\pi}{i}}\int_0^1 dx \int_0^\infty \frac{d\beta}{\beta^{1/2}}\left[(1+x)^2p^2 + \frac{3i}{2\beta}\right]e^{i\beta b}, \tag{B.14}$$

with $b = p^2x(1-x) - m^2(1-x) - M^2(1-x)$. Combining (B.13) and (B.14) to calculate the contribution of the diagram 3.4(a) to the propagator in scalar QED and performing the x and β integrations, we obtain

$$-i\Sigma_{\text{reg}}^{3.4a}(p) = \frac{ie^2}{2\pi}m\left[\left(\ln\left(\frac{\mu}{m}\right) - \ln(2) + \frac{1}{2}\right) + \frac{M}{2m}\right]. \tag{B.15}$$

We can also calculate the tadpole diagram 3.4(b) which, after some algebra, yields

$$-i\Sigma_{\text{reg}}^{3.4b}(p) = \frac{3ie^2}{4\pi}M. \tag{B.16}$$

Adding (B.15) and (B.16), we obtain

$$-i\Sigma_{\text{reg}}(p) = \frac{ie^2}{2\pi}\left[\left(\ln\left(\frac{\mu}{m}\right) - \ln(2) + \frac{1}{2}\right) + \frac{M}{m}\right], \tag{B.17}$$

which is the complete expression for the one loop self-energy in Feynman gauge.

Now we can use the renormalisation conditions outlined in (3.86) to obtain

$$\delta m = \frac{e^2}{4\pi}\left[\left(\ln\left(\frac{\mu}{m}\right) - \ln(2) + \frac{1}{2}\right) + \frac{M}{m}\right], \tag{B.18}$$

and

$$\delta Z_2 = \frac{e^2}{4\pi}\frac{1}{m}\left[\frac{1}{\mu} + \frac{1}{2}\right]. \tag{B.19}$$

The UV divergence corresponds to the term M/m in the mass shift. Notice that the IR divergences, as well as the finite terms associated with the mass shift and the wave function renormalisation constant, are unchanged. Also note that there is no UV divergence in the wave function renormalisation in 2+1 dimensional scalar QED.

Appendix C

The Dressing Equation from Heavy Matter

In Chapter 2 we discussed how to construct the dressing equation from the asymptotic interaction Hamiltonian. This appendix is based upon the study of a heavy matter field [7, 12]. We shall, in particular, consider scalar QED as it is well known [62] that the IR structures in QED are independent of the spin of the matter particles. To proceed any further, let us recall the matter part of the QED Lagrangian in scalar theory

$$\mathcal{L}^{\text{matter}} = (D_\mu\phi)^\dagger(D_\mu\phi) - m^2\phi^\dagger\phi\,, \tag{C.1}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative.

In order to use the infinite mass limit [80] of the scalar field, we need to specify the mass-shell point of the particle. We therefore introduce the rescaled fields [7]

$$\tilde{\phi}(x) := \sqrt{2m}e^{imu\cdot x}\phi(x)\,, \tag{C.2}$$

where u describes the four-velocity of the heavy particle. In terms of the new fields, the matter part of the Lagrangian becomes

$$\mathcal{L}^{\text{matter}} = \frac{1}{2m}(D_\mu\tilde{\phi})^\dagger(D_\mu\tilde{\phi}) + i\tilde{\phi}^\dagger u^\mu D_\mu\tilde{\phi}\,, \tag{C.3}$$

In the large m limit only the first term survives and the equations of motion for this part become

$$u^\mu D_\mu\tilde{\phi} = 0 \tag{C.4}$$

To construct a gauge invariant heavy charged particle, we define as in Chapter 2

$$\tilde{\Phi}(x) = h^{-1}(x)\tilde{\phi}(x)\,. \tag{C.5}$$

Under local gauge transformation h^{-1} is transformed as (2.47). Substituting this dressed field into (C.4), we find that

$$u\cdot\partial h^{-1}(x) = -ieh^{-1}(x)u\cdot A(x)\,, \tag{C.6}$$

This is the *dressing equation* and it is exactly the same as (2.53). As in Chapter 2, we can solve (C.6) together with (2.47), to find the desired form of the dressing.

Appendix D

Dimensional Regularisation and Power Divergences

The aim of this appendix is to study the effect of dimensional regularisation on power divergences. We consider the example of massive tadpoles in various dimensions.

The following standard Euclidean integral appears in many textbook discussions of dimensional regularisation (with $D = 4 - 2\epsilon$)

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = -\frac{m^2}{(4\pi)^2} \frac{1}{\varepsilon} + \text{finite}\,. \tag{D.1}$$

This is an ultra-violet divergence since the integral is finite in the infra-red region. Upon reflection this seems to be an unusual result for two reason. First the RHS is negative but the left hand side is positive. Second, by power counting, the integral must be quadratically divergent though it appears to be logarithmically divergent (a power of m^2 ensures that the dimension is correct).

If, instead, we carry out the integral using a symmetric UV Euclidean cutoff,

$k^2 \leq \Lambda^2$, we would obtain for the same integral

$$\int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} = \frac{1}{(4\pi)^2}\left[\Lambda^2 - m^2 \ln(\Lambda^2)\right] + \text{finite}. \tag{D.2}$$

Comparing these two results we see that dimensional regularisation has set the leading, quadratic divergence to zero and regulated the subleading logarithmic divergence according to the dictionary

$$\ln(\Lambda^2) \leftrightarrow \frac{1}{\epsilon}. \tag{D.3}$$

The minus sign in front of $1/\epsilon$ in (D.1) can be understood as coming from the expansion in the UV region of $1/(k^2+m^2) = 1/k^2 - m^2/k^4 + \mathcal{O}(1/k^6)$.

It is easy to verify that this 'dictionary' generally holds. Consider for example the logarithmically divergent integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)^2} = \frac{1}{16\pi^2}\frac{1}{\varepsilon} + \text{finite}, \tag{D.4}$$

or the quartically divergent

$$\int \frac{d^6k}{(2\pi)^6} \frac{1}{k^2+m^2} = \frac{1}{128\pi^3}\frac{1}{\varepsilon} + \text{finite}. \tag{D.5}$$

In both cases it can be checked that the result of dimensional regularisation for the integrals corresponds exactly to the logarithmic part of the result obtained using the direct cutoff and the above dictionary.

Finally, consider the integral

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2+m^2}, \tag{D.6}$$

which by power counting has a linear divergence in the UV domain ($= \frac{1}{2\pi}\Lambda$). By expanding the denominator in powers of $1/k^2$ it can be shown that the integral is finite apart from this leading linear divergence (i.e., there is no logarithmic singularity). This integral is indeed also found to be *finite* in dimensional regularisation. More on this and various responses can be found in Section 3.1

Appendix E

About the Integrals

In this appendix, we will present some important formulae which are needed to calculate covariant and non-covariant integrals. Using these, we then calculate two important non-covariant integrals that arises in Coulomb gauge calculations for the electron propagator in Chapter 2. A treatment of such integrals was first considered by Adkins in [63] and here we follow his basic technique.

We first write all the useful formulae that we repeatedly use in Chapter 2 and 3. Formulae for covariant integrals with dimensional regularisation are as follows:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - a)^\alpha} = \frac{i(-1)^\alpha \pi^{D/2}}{(2\pi)^D} \frac{\Gamma\left(\alpha - \frac{D}{2}\right)}{\Gamma(\alpha)} \frac{1}{a^{\left(\alpha - \frac{D}{2}\right)}}, \tag{E.1}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - a)^\alpha} = -\frac{i(-1)^\alpha \pi^{D/2}}{(2\pi)^D} \frac{D}{2} \frac{\Gamma\left(\alpha - 1 - \frac{D}{2}\right)}{\Gamma(\alpha)} \frac{1}{a^{\left(\alpha - 1 - \frac{D}{2}\right)}}, \tag{E.2}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - a)^\alpha} = -\frac{i(-1)^\alpha (\pi)^{D/2}}{(2\pi)^D} \frac{g_{\mu\nu}}{2} \frac{\Gamma\left(\alpha - 1 - \frac{D}{2}\right)}{\Gamma(\alpha)} \frac{1}{a^{\left(\alpha - 1 - \frac{D}{2}\right)}}, \tag{E.3}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k_\mu}{(k^2 - a)^\alpha} = 0, \tag{E.4}$$

where $a = m^2 x - p^2 x(1 - x)$

Formulae for non-covariant integrals with dimensional regularisation are as follows:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{(k^2 - 2k\cdot p - M^2)^\alpha} \frac{B}{[k^2 - (k\cdot\eta)^2]^\beta}$$
$$= \frac{(-1)^{\alpha+\beta}}{(2\pi)^D} \frac{i\pi^{D/2}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{dx}{\sqrt{1-x}} (1-x)^{\alpha-1} x^{\beta-1} C\,, \tag{E.5}$$

where various B's and C's are related as follows:

$$B = 1\,, \qquad C = \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{a_g^{\alpha+\beta-\frac{D}{2}}}\,,$$
$$B = k_\mu\,, \qquad C = (1-x)(Ap)_\mu \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{a_g^{\alpha+\beta-\frac{D}{2}}}\,,$$
$$B = k_\mu k_\nu, \quad C = (1-x)^2 (Ap)_\mu (Ap)_\nu \frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)}{a_g^{\alpha+\beta-\frac{D}{2}}}$$
$$-\frac{1}{2} A_{\mu\nu} \frac{\Gamma\left(\alpha+\beta-1-\frac{D}{2}\right)}{a_g^{\alpha+\beta-1-\frac{D}{2}}}\,. \tag{E.6}$$

where

$$A_{\mu\nu} = g_{\mu\nu} + \frac{x}{1-x}\eta_\mu\eta_\nu\,, \tag{E.7}$$

and we have also introduced the notation

$$a_g = (1-x)[(1-x)A_{\mu\nu}p^\mu p^\nu + M^2]\,. \tag{E.8}$$

For integrals occurring in the Coulomb gauge calculations of (2.86) this is generalised as follows. If we have one covariant and one non-covariant denominator we have for a_g

$$a_2 = (1-x)[\Pi + m^2 - p^2] \tag{E.9}$$

If we have two covariant and one covariant denominators then a_g takes the form

$$a_3 = u(1-x)\{u\Pi + m^2 - p^2\}\,, \tag{E.10}$$

where we have further introduced the notation

$$\Pi = (1-x)p^2 + x(p\cdot\eta)^2\,, \tag{E.11}$$

and u is the usual Feynman parameter when we combine two covariant denominators.

Using these formulae we can calculate two non-covariant integrals, which will be useful for the mass shell renormalisation of the Coulomb gauge propagator in Chapter 2.

We need the integrals of the form

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D}\,\frac{1}{(p-k)^2-m^2}\,\frac{k_\mu}{k^2-(k\cdot\eta)^2} = I_1^p p_\mu + p\cdot\eta I_1^\eta \eta_\mu\,, \tag{E.12}$$

where we find for on-shell momentum, p

$$\begin{aligned}
\tilde{I}_1^p &= \frac{i}{16\pi^2}\int_0^1 \frac{dx}{\sqrt{1-x}}\,(1-x)\left[\frac{1}{\hat{\varepsilon}} - \ln\left((1-x)^2 + \frac{(p\cdot\eta)^2}{m^2}x(1-x)\right)\right]\,,\\
\tilde{I}_1^\eta &= \frac{i}{16\pi^2}\int_0^1 \frac{dx}{\sqrt{1-x}}\,x\left[\frac{1}{\hat{\varepsilon}} - \ln\left((1-x)^2 + \frac{(p\cdot\eta)^2}{m^2}x(1-x)\right)\right]\,.
\end{aligned} \tag{E.13}$$

We also need the integrals

$$\begin{aligned}
\int \frac{\mathrm{d}^D k}{(2\pi)^D}\,\frac{1}{k^2}\,\frac{1}{(p-k)^2-m^2}\,\frac{k_\mu k_\nu}{k^2-(k\cdot\eta)^2} &= I_2^g g_{\mu\nu} + I_2^\eta \eta_\mu\eta_\nu + I_2^{pp} p_\mu p_\nu + (p\cdot\eta)^2 I_2^{\eta\eta}\eta_\mu\eta_\nu\\
&+\ p\cdot\eta I_2^{p\eta}(p_\mu\eta_\nu + \eta_\mu p_\nu)\,,
\end{aligned} \tag{E.14}$$

where

$$
\begin{aligned}
\tilde{I}_2^g &= \frac{1}{2}\tilde{I}_1^p + \frac{i}{16\pi^2}\frac{2}{3}, \\
\tilde{I}_2^\eta &= \frac{1}{2}\tilde{I}_1^\eta + \frac{i}{16\pi^2}\frac{4}{3}, \\
\tilde{I}_2^{pp} &= \frac{i}{16\pi^2}\int_0^1 \frac{dx}{\sqrt{1-x}}\frac{-(1-x)^2}{\tilde{\Pi}}, \\
\tilde{I}_2^{\eta\eta} &= \frac{i}{16\pi^2}\int_0^1 \frac{dx}{\sqrt{1-x}}\frac{-x^2}{\tilde{\Pi}}, \\
\tilde{I}_2^{p\eta} &= \frac{i}{16\pi^2}\int_0^1 \frac{dx}{\sqrt{1-x}}\frac{-x(1-x)}{\tilde{\Pi}},
\end{aligned}
\tag{E.15}
$$

where a tilde signifies that the function is evaluated at an arbitrary point on the mass shell.

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