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Dedicated to the memory of K. R. Parthasarathy, master of probability and a great inspiration to us both

Abstract

The method of Feynman–Kac perturbation of quantum stochastic processes has a long pedigree, with the theory usually developed within the framework of processes on von Neumann algebras. In this work, the theory of operator spaces is exploited to enable a broadening of the scope to flows on C^* algebras. Although the hypotheses that need to be verified in this general setting may seem numerous, we provide auxiliary results that enable this to be simplified in many of the cases which arise in practice. A wide variety of examples is provided by way of illustration.

1 Introduction

Very early in the study of quantum stochastic processes on operator algebras, it was realised in pioneering work of Accardi [1] that analogues of the method of Feynman–Kac perturbation in classical probability could be used to perturb Markov semigroups with cocycles to give new semigroups, giving meaning to a formal sum of generators.

The creation of quantum stochastic calculus in the early 1980s gave a means of constructing such a cocycle j on a $*$-algebra $\mathsf{A} \subseteq B(\mathsf{h})$ by solving the quantum stochastic differential equation

$$j_0(x) = x \otimes I_{\mathcal{F}} \qquad \text{and} \qquad \mathrm{d}j_t(x) = \widetilde{\jmath}_t\big(\phi(x)\big)\,\mathrm{d}\Lambda_t \qquad (x \in \mathsf{A}_0 \subseteq \mathsf{A}), \tag{1.1}$$

the cocycle in this case being used to perturb the ampliated CCR flow σ on Boson Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$. Here h and k are known as the initial space and multiplicity space, respectively, and A_0 is a dense, unital $*$-subalgebra of A. Solving this QSDE is possible for any completely bounded generator ϕ [21], and this gives a cocycle in a generalised sense, but for many purposes it is desirable that j be $*$-homomorphic. Conversely, a completely positive and contractive cocycle j on a C^* algebra A that satisfies the additional continuity requirement of being *Markov regular* or *elementary* necessarily satisfies (1.1), in which case the stochastic generator ϕ must be completely bounded [19, 20]; see also [23, Theorem 6.4].

Consequently it has become a folklore result that solutions of the QSDE (1.1) and cocycles should be essentially the same thing. However, as soon as one loosens the restrictive continuity or boundedness assumptions, the correspondence between solutions of (1.1) and cocycles is a lot less clear, especially because of the many difficulties inherent in solving (1.1) for an unbounded generator ϕ. Our previous work [7] gave a method for solving (1.1) on a C^* algebra for certain unbounded stochastic generators, with the solution being $*$-homomorphic and a cocycle. This involved two assumptions: a type of domain invariance for ϕ and growth estimates for the iterates of ϕ whose existence follows from the first assumption. In that paper we gave a range of interesting examples where these assumptions held, but these assumptions are obviously still somewhat restrictive.

In this paper we now follow the work of Evans and Hudson [13] for perturbing a cocycle j, called the free flow, of the CCR flow σ to give a new cocycle k for σ. This involves finding bounded solutions to the multiplier equation, which is the following operator-valued QSDE with time-dependent coefficients:

$$X_0 = I_{\mathsf{h}\otimes\mathcal{F}} \qquad \text{and} \qquad \mathrm{d}X_t = \widetilde{\jmath}_t(F)\widetilde{X}_t \,\mathrm{d}\Lambda_t. \tag{1.2}$$

The multiplier generator $F \in \mathsf{A} \otimes B(\widehat{\mathsf{k}})$, where $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$. The cocycle k is then given by setting $k_t(x) := X_t^* j_t(x) X_t$. The process obtained by conjugation with X is, at least formally, a cocycle of the semigroup $J = (\widehat{\jmath}_t \circ \sigma_t)_{t\in\mathbb{R}_+}$; see Section 3 for full details.

Evans and Hudson worked with bounded ϕ for an arbitrary algebra A and with finite-dimensional multiplicity space k. Their work was extended to the case where k is separable and infinite dimensional, A is a von Neumann algebra and ϕ is completely bounded in [11] and [15]. A more thorough analysis was later given in [5], [6] and [7], where A is still a von Neumann algebra and either ϕ is completely bounded or h and k are separable. This included an algebraic characterisation of solutions to (1.2).

Here, we move to the broader setting of C^* algebras. The technical difficulties faced are much greater than before. Above, we were vague about the nature of the tensor-product symbol. For those prior works where A is a von Neumann algebra it will be the von Neumann tensor product and all bounded maps will be required to be normal. For processes on C^* algebras this must be modified; we have to use matrix-space tensor products such as $\mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$, and these need not produce algebras [21]. Moreover, the matrix-space liftings of $*$-homomorphisms such as $j_t \otimes_{\mathrm{m}} \mathrm{id}_{B(\widehat{\mathsf{k}})}$ need not preserve products. Thus much greater care is needed as, for example, checking equalities for simple tensors and extending by linearity and continuity will no longer suffice.

This paper is structured as follows. Section 2 contains the necessary definitions and results concerning matrix spaces. Section 3 introduces and gives general results about quantum stochastic processes, cocycles and the QSDE (1.1). Here we work with processes on a general operator space V; this point of view was used to unify several results about various forms of cocycles in [22] and [23]. The key result is Theorem 3.15 which gives conditions under which weak solutions of (1.1) are actually cocycles. Section 4 contains our main results: after introducing the notion of a free flow j, now on a C^* algebra, and solving the corresponding multiplier equation, we establish the effects of Feynman–Kac perturbation. We have given the result in a very general form, necessitating a long list of hypotheses regarding multiplicativity of liftings and measurability of processes, but also provide a number of auxiliary results that can be used to simplify matters greatly under conditions that frequently arise in practice. In Section 5, these results are applied to a wide variety of examples. Our previous work on the quantum exclusion process from [7] is extended; there we obtained the growth estimates by assuming a symmetry condition on the amplitudes, which can now be relaxed by means of the perturbation techniques of this paper. (An alternative construction of exclusion processes has been given in [23] by making use of the semigroup characterisation of completely positive and contractive cocycles given in [22] that generalised earlier work of Accardi and Kozyrev [2].) Similarly our previous work on flows on universal C^* algebras from [7] is extended; Feynman–Kac perturbations had already been used in this context for flows on the non-commutative torus in [9]. Here we show that those techniques apply more generally for other algebras such as the Cuntz algebras and the non-commutative spheres of Banica [3].

1.1 Conventions and notation

The indicator function of a set A is denoted 1_A (with its domain being clear from the context) whereas 1_{A} is the multiplicative identity for the unital C^* algebra A and $\mathbb{1}_P$ equals 1 if the proposition P is true and 0 if it is false. All vector spaces have complex scalar field; all inner products are linear in their second argument. The identity operator on a vector space X is denoted I_X whereas the identity operator on an operator space V is denoted id_{V}. The Banach algebra of bounded operators on a Banach space X is denoted $B(X)$; the C^* algebra of $n \times n$ matrices with entries from a C^* algebra A is denoted $M_n(\mathsf{A})$. Algebraic, spatial and ultraweak tensor products are denoted $\underline{\otimes}$, $\otimes$ and $\overline{\otimes}$, respectively. The sets of non-negative real numbers and integers are denoted $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$; the set of natural numbers is denoted $\mathbb{N} := \{1, 2, 3, \ldots\}$.

2 Matrix spaces and liftings

Definition 2.1. Given Hilbert spaces h and H and a vector $z \in \mathsf{H}$, let $E_z \in B(\mathsf{h}; \mathsf{h} \otimes \mathsf{H})$ be such that $E_z u = u \otimes z$ for all $u \in \mathsf{h}$ and let $E^z := E_z^* \in B(\mathsf{h} \otimes \mathsf{H}; \mathsf{h})$ be its adjoint. Using Dirac notation, we may write $E_z = I_{\mathsf{h}} \otimes |z\rangle$ and $E^z = I_{\mathsf{h}} \otimes \langle z|$.

Definition 2.2. Let $\mathsf{V} \subseteq B(\mathsf{h})$ be a concrete operator space, that is, a norm-closed linear subspace of $B(\mathsf{h})$. Given any Hilbert space H, the *matrix space over* V,

$$\mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H}) := \{T \in B(\mathsf{h} \otimes \mathsf{H}) : E^z T E_w \in \mathsf{V} \text{ for all } z, w \in \mathsf{H}\},$$

is an operator space such that $\mathsf{V} \otimes B(\mathsf{H}) \subseteq \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H}) \subseteq \mathsf{V} \mathbin{\overline{\otimes}} B(\mathsf{H})$. The first inclusion is an equality if H is finite dimensional; the latter inclusion is an equality if and only if V is ultraweakly closed. If K is another Hilbert space then $\big(\mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H})\big) \otimes_{\mathrm{m}} B(\mathsf{K}) = \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H} \otimes \mathsf{K})$ with the standard identifications.

Rectangular matrix spaces of the form $\mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H}; \mathsf{K})$ are defined in the same way. Two important examples are the *column space*

$$\mathsf{V} \otimes_{\mathrm{m}} |\mathsf{H}\rangle := \{T \in B(\mathsf{h}; \mathsf{h} \otimes \mathsf{H}) : E^z T \in \mathsf{V} \text{ for all } z \in \mathsf{H}\} = \mathsf{V} \otimes_{\mathrm{m}} B(\mathbb{C}; \mathsf{H})$$

and the *row space*

$$\mathsf{V} \otimes_{\mathrm{m}} \langle \mathsf{H}| := \{T \in B(\mathsf{h} \otimes \mathsf{H}; \mathsf{h}) : TE_w \in \mathsf{V} \text{ for all } w \in \mathsf{H}\} = \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H}; \mathbb{C}).$$

Definition 2.3. If $\Phi : \mathsf{V} \to \mathsf{W}$ is a completely bounded map between operator spaces and H is a Hilbert space then the *matrix-space lifting* of Φ is the unique completely bounded map

$$\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})} : \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H}) \to \mathsf{W} \otimes_{\mathrm{m}} B(\mathsf{H})$$

such that $E^z(\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(T)E_w = \Phi(E^z T E_w)$ for all $T \in \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H})$ and $z, w \in \mathsf{H}$. It then holds that $\|\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})}\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{cb}}$. For the existence of such a map, see [21] or [4, Theorem 2.5].

If K is another Hilbert space then, with the standard identifications,

$$(\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})}) \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{K})} = \Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H} \otimes \mathsf{K})}. \tag{2.1}$$

Lemma 2.4. *If $\Phi : \mathsf{A} \to \mathsf{B}$ is a completely positive map between unital C^* algebras and K is a Hilbert space then the matrix-space lifting $\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{K})}$ is a completely positive map between operator systems.*

Proof. Assume without loss of generality that $\mathsf{B} \subseteq B(\mathsf{H})$ and let $T \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{K})$ be positive. To prove that $(\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{K})})(T)$ is positive, it suffices to show that

$$\langle (I_{\mathsf{H}} \otimes P)\xi, (\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{K})})(T)(I_{\mathsf{H}} \otimes P)\xi \rangle \geqslant 0 \qquad (\xi \in \mathsf{H} \otimes \mathsf{K})$$

for any finite-rank orthogonal projection $P \in B(\mathsf{K})$. If P has rank n then $(I_{\mathsf{H}} \otimes P)\xi$ and PTP may be considered to be elements of H^n and $M_n(\mathsf{A})$, respectively, with the latter being positive, so that

$$\langle (I_{\mathsf{H}} \otimes P)\xi, (\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{K})})(T)(I_{\mathsf{H}} \otimes P)\xi \rangle = \langle (I_{\mathsf{H}} \otimes P)\xi, (\Phi \otimes_{\mathrm{m}} \mathrm{id}_{\mathbb{C}^n})(PTP)(I_{\mathsf{H}} \otimes P)\xi \rangle \geqslant 0,$$

as required. Since K is arbitrary, complete positivity now follows by (2.1). □

3 Quantum stochastic processes, cocycles and differential equations

3.1 Stochastic processes

Notation 3.1. Fix a Hilbert space k, the *multiplicity space*. For any subinterval J of $\mathbb{R}_+$, let $\mathcal{F}_J$ denote the Boson Fock space over $L^2(J; \mathsf{k})$. The exponential vector corresponding to $g \in L^2(J; \mathsf{k})$ is $\varepsilon(g) = \big((n!)^{-1/2} g^{\otimes n}\big)_{n \in \mathbb{Z}_+}$. For brevity, let $\mathcal{F} := \mathcal{F}_{\mathbb{R}_+}$, $\mathcal{F}_{t)} := \mathcal{F}_{[0,t)}$ and $\mathcal{F}_{[t} := \mathcal{F}_{[t,\infty)}$ for all $t \in \mathbb{R}_+$, with similar abbreviations for the identity operators I, $I_{t)}$ and $I_{[t}$ on these spaces, and where $\mathcal{F}_{0)} := \mathbb{C}$. Recall the tensor-product decomposition

$$\mathcal{F} \cong \mathcal{F}_{t)} \otimes \mathcal{F}_{[t};\ \varpi(f) \leftrightarrow \varpi(f|_{[0,t)}) \otimes \varpi(f|_{[t,\infty)}) \qquad \text{for all } t \in \mathbb{R}_+, f \in L^2(\mathbb{R}_+; \mathsf{k}),$$

where $\varpi(g) = \exp(-\frac{1}{2}\|g\|^2)\varepsilon(g)$ is the normalised exponential vector corresponding to g. This identification will be used frequently without comment.

The *extended multiplicity space* is $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, and $\widehat{z} := (1, z) \in \widehat{\mathsf{k}}$ for all $z \in \mathsf{k}$.

Let $\Delta_{\mathsf{H}} := I_{\mathsf{H}} \otimes P_{\mathsf{k}} \in B(\mathsf{H} \otimes \widehat{\mathsf{k}})$, where $P_{\mathsf{k}} \in B(\widehat{\mathsf{k}})$ is the orthogonal projection onto $\{0\} \oplus \mathsf{k}$. This will be abbreviated to Δ when the Hilbert space H is clear from the context.

Definition 3.2. An *admissible set* is a subset of k which contains 0 and is total in k. For such a set T, let $L^{\text{step}}(\mathbb{R}_+; \mathsf{T}) = \operatorname{lin}\{1_{[0,t)}x : x \in \mathsf{T}, t > 0\}$ denote the set of T-valued right-continuous step functions on $\mathbb{R}_+$. Admissibility of T ensures that $\mathcal{E}(\mathsf{T})$, the linear span of the exponential vectors corresponding to elements of $L^{\text{step}}(\mathbb{R}_+; \mathsf{T})$, is dense in $\mathcal{F}$ [16, Proposition 2.1].

Definition 3.3. Fix a Hilbert space h, the *initial space*. For a fixed admissible set T, an *operator process* X is a collection of linear operators $(X_t)_{t \in \mathbb{R}_+}$ such that

(i) $\mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \subseteq \operatorname{dom} X_t \subseteq \mathsf{h} \otimes \mathcal{F}$ and $\operatorname{im} X_t \subseteq \mathsf{h} \otimes \mathcal{F}$ for all $t \in \mathbb{R}_+$,

(ii) adaptedness holds, in the sense that

$$E^{\varpi(f)} X_t E_{\varpi(g)} = \langle \varpi(1_{[t,\infty)} f), \varpi(1_{[t,\infty)} g) \rangle E^{\varpi(1_{[0,t)} f)} X_t E_{\varpi(1_{[0,t)} g)}$$

for all f, $g \in L^{\text{step}}(\mathbb{R}_+; \mathsf{T})$ and $t \in \mathbb{R}_+$, and

(iii) $t \mapsto X_t \xi$ is weakly measurable for all $\xi \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T})$.

The operator process X is *strongly measurable* or *strongly continuous* if $t \mapsto X_t \xi$ is strongly measurable or norm continuous, respectively, for all $\xi \in \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T})$; it is *weakly continuous* if $t \mapsto \langle \zeta, X_t \xi \rangle$ for all ζ, $\xi \in \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T})$. The operator process X is *bounded*, *contractive*, *isometric*, *co-isometric* or *unitary* if each operator X_t has this property. In these cases we will automatically identify X_t with its continuous extension to all of $\mathsf{h} \otimes \mathcal{F}$, and then adaptedness means that $X_t = X_{t)} \otimes I_{[t,\infty)}$ for some $X_{t)} \in B(\mathsf{h} \otimes \mathcal{F}_{t)})$.

Remark 3.4. (i) A bounded operator process which is strongly continuous has locally bounded norm if and only if it is strongly continuous on all of $\mathsf{h} \otimes \mathcal{F}$, by the Banach–Steinhaus Theorem.

(ii) If X and Y are strongly continuous bounded operator processes that each have locally bounded norm then the product $XY = (X_t Y_t)_{t \in \mathbb{R}_+}$ is also a strongly continuous operator process with locally bounded norm.

(iii) If h and k are both separable then all operator processes are strongly measurable, by Pettis' Theorem.

Definition 3.5. A *completely bounded mapping process* k on an operator space $\mathsf{V} \subseteq B(\mathsf{h})$ is a collection of completely bounded maps $\big(k_t : \mathsf{V} \to B(\mathsf{h} \otimes \mathcal{F})\big)_{t \in \mathbb{R}_+}$ such that $\big((k_t(x)\big)_{t \in \mathbb{R}_+}$ is an operator process for each $x \in \mathsf{V}$. The adaptedness of each of these processes implies for each $t \in \mathbb{R}_+$ the existence of a completely bounded map $k_{t)} : \mathsf{V} \to B(\mathsf{h} \otimes \mathcal{F}_{t)})$ such that $k_t(x) = k_{t)}(x) \otimes I_{[t}$ for all $x \in \mathsf{V}$.

If k is a completely bounded mapping process on V then k is *strongly measurable* or *strongly continuous* if the bounded operator process $\big(k_t(x)\big)_{t\in\mathbb{R}_+}$ has the same property for all $x \in \mathsf{V}$. If V is a $*$-algebra then the mapping process k is *$*$-homomorphic* if k_t is a $*$-homomorphism for each $t \in \mathbb{R}_+$.

3.2 Cocycles

Definition 3.6. Given a completely bounded mapping process k on V, let

$$k_t[f,g] : \mathsf{V} \to B(\mathsf{h});\ x \mapsto E^{\varpi(1_{[0,t)}f)} k_t(x) E_{\varpi(1_{[0,t)}g)} = E^{\varpi(f|_{[0,t)})} k_{t)}(x) E_{\varpi(g|_{[0,t)})}$$

for all $t \in \mathbb{R}_+$ and $f,\ g \in L^2(\mathbb{R}_+;\mathsf{k})$. A *Markovian cocycle on* V is a completely bounded mapping process k such that, for some admissible set T,

(i) $k_t[f,g](x) \in \mathsf{V}$ for all $t \in \mathbb{R}_+$, $f,\ g \in L^{\text{step}}(\mathbb{R}_+;\mathsf{T})$ and $x \in \mathsf{V}$,

(ii) $k_0[0,0] = \mathrm{id}_{\mathsf{V}}$ and

(iii) $k_{s+t}[f,g] = k_s[f,g] \circ k_t[f(\cdot+s), g(\cdot+s)]$ for all $s,\ t \in \mathbb{R}_+$ and $f,\ g \in L^{\text{step}}(\mathbb{R}_+;\mathsf{T})$.

Since k is adapted, if (ii) holds then $k_0[f,g] = \mathrm{id}_{\mathsf{V}}$ for all $f,\ g$ in $L^{\text{step}}(\mathbb{R}_+;\mathsf{T})$.

Remark 3.7. Totality of the set $\{\varpi(f) : f \in L^{\text{step}}(\mathbb{R}_+;\mathsf{T})\}$, linearity and norm continuity of the map $z \mapsto E_z$, and adaptedness of k, show that condition (i) of Definition 3.6 is equivalent to the requirement that

(i)$'$ $k_t(\mathsf{V}) \subseteq \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})$.

Furthermore, in this formulation, the following equivalent versions of conditions (ii) and (iii) appear as well:

(ii)$'$ $k_0(x) = x \otimes I_{\mathcal{F}}$ for all $x \in \mathsf{V}$ and

(iii)$'$ $k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t$ for all $s, t \in \mathbb{R}_+$,

where $\sigma_s : B(\mathsf{h} \otimes \mathcal{F}) \to B(\mathsf{h} \otimes \mathcal{F}_{[s})$ is the ampliated right shift, arising from the natural unitary identification $\mathsf{h} \otimes \mathcal{F} \cong \mathsf{h} \otimes \mathcal{F}_{[s}$. In particular, if $T \in B(\mathsf{h} \otimes \mathcal{F})$ then

$$E^{\varpi(f)} \sigma_s(T) E_{\varpi(g)} = E^{\varpi(f(\cdot+s))} T E_{\varpi(g(\cdot+s))} \qquad \text{for all } f, g \in L^2\big([s,\infty);\mathsf{k}\big),$$

and $\widehat{k}_s := k_{s)} \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathcal{F}_{[s})}$; note the equality and inclusion

$$\sigma_s\big(\mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})\big) = \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F}_{[s}) \qquad \text{and} \qquad k_{s)}(\mathsf{V}) \subseteq \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F}_{s)}),$$

where the latter follows from condition (i). The definition of σ_s here differs slightly from that used in [22].

Thus whether a completely bounded mapping process is a Markovian cocycle is independent of the choice of admissible set T, and the set $L^{\text{step}}(\mathbb{R}_+;\mathsf{T})$ in Definition 3.6 can be replaced by any subset S of $L^2(\mathbb{R}_+;\mathsf{k})$ for which $\{\varpi(f) : f \in S\}$ is total in $\mathcal{F}$.

The following proposition is the matrix-space version of the corresponding result for normal mapping processes on von Neumann algebras [6, Lemma 1.6(b)].

Proposition 3.8. *Let k be a completely bounded mapping process on V. Then k is a Markovian cocycle if and only if $(K_t := \widehat{k}_t \circ \sigma_t)_{t\in\mathbb{R}_+}$ is a semigroup on $\mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})$, that is,*

$$K_t\big(\mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})\big) \subseteq \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F}), \quad K_0 = \mathrm{id}_{\mathsf{V}\otimes_{\mathrm{m}} B(\mathcal{F})} \quad \textit{and} \quad K_{s+t} = K_s \circ K_t \quad \textit{for all } s, t \in \mathbb{R}_+.$$

Proof. Note first that $\operatorname{dom} K_t = \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})$, and that

$$k_t(x) = K_t(x \otimes I) \qquad \text{for all } t \in \mathbb{R}_+ \text{ and } x \in \mathsf{V};$$

it follows immediately that if K is a semigroup on $\mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})$ then k is a Markovian cocycle. For the converse, let k be a completely bounded mapping process on V. The identity

$$E^{\varpi(f)} K_s(T) E_{\varpi(g)} = k_s[f, g]\big(E^{\varpi(f(\cdot+s))} T E_{\varpi(g(\cdot+s))}\big)$$

holds for all $f, g \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$, $s \in \mathbb{R}_+$ and $T \in \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})$. Hence if k is a Markovian cocycle then $K_s\big(\mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})\big) \subseteq \mathsf{V} \otimes_{\mathrm{m}} B(\mathcal{F})$. Moreover, this identity and conditions (ii) and (iii) of Definition 3.6 then give that

$$K_0(T) = T \qquad \text{and} \qquad K_s\big(K_t(T)\big) = K_{s+t}(T)$$

for all s, $t \in \mathbb{R}_+$ as required. □

3.3 Semigroup representation

Notation 3.9. Given a completely bounded mapping process k on V, let

$$\mathcal{P}_t^{z,w} := k_t[1_{[0,t)} z, 1_{[0,t)} w] : \mathsf{V} \to B(\mathsf{h});\ x \mapsto E^{\varpi(1_{[0,t)} z)} k_t(x) E_{\varpi(1_{[0,t)} w)}$$

for all z, $w \in \mathsf{k}$ and $t \in \mathbb{R}_+$.

Theorem 3.10. *A completely bounded mapping process k on V is a Markovian cocycle if and only if there exists an admissible set T such that $(\mathcal{P}_t^{z,w})_{t\in\mathbb{R}_+}$ is a semigroup on V for all z, $w \in \mathsf{T}$ and, given any f, $g \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ subordinate to some partition $\{0 = t_0 < \cdots < t_n < \cdots\}$ of $\mathbb{R}_+$, it holds that*

$$k_t[f, g] = \mathcal{P}_{t_1 - t_0}^{z_0, w_0} \circ \cdots \circ \mathcal{P}_{t - t_n}^{z_n, w_n} \qquad \textit{whenever } t \in [t_n, t_{n+1}), \tag{3.1}$$

where $z_j = f(t_j)$ and $w_j = g(t_j)$ for $j = 0, \ldots, n$. In this case the decomposition (3.1) holds for any choice of admissible set T.

Proof. See [22, Proposition 5.1]. □

Remark 3.11. Let k be a completely bounded mapping process on V and let T be an admissible set. If $\{\mathcal{P}_t^{z,w} : z, w \in \mathsf{T},\ t \in \mathbb{R}_+\}$ is a family of linear maps on V that satisfies (3.1) for all functions $f,\ g \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ that are subordinate to the partition $\{0 = t_0 < \cdots < t_n < \cdots\}$ then

$$\mathcal{P}_s^{z,w} \circ \mathcal{P}_t^{z,w} = \mathcal{P}_{s+t}^{z,w} \qquad \text{for all } z, w \in \mathsf{T} \text{ and } s, t \in \mathbb{R}_+.$$

To see this, let f and g take the values z and w, respectively, on a sufficiently large interval containing the origin. Thus, it is not necessary to verify independently that the semigroup property holds when applying Theorem 3.10 to produce a Markovian cocycle.

3.4 Differential equations

Notation 3.12. Let V_0 be a subset of V and let $\psi : \mathsf{V}_0 \to \mathsf{V} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ be a map. Given a completely bounded mapping process k on V define $\widetilde{k}_t := k_t \otimes_{\mathrm{m}} \mathrm{id}_{B(\widehat{\mathsf{k}})}$. For a fixed admissible set T, the statement that

$$\mathrm{d}k_t(x) = (\widetilde{k}_t \circ \psi)(x)\, \mathrm{d}\Lambda_t \qquad \text{weakly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T})$$

for some $x \in \mathsf{V}_0$ means that

$$t \mapsto \langle u \otimes \varpi(f), k_t\big(E^{\widehat{f(t)}} \psi(x) E_{\widehat{g(t)}}\big) v \otimes \varpi(g)\rangle$$

is locally integrable and

$$\langle u \otimes \varpi(f), \big(k_t(x) - k_0(x)\big) v \otimes \varpi(g)\rangle = \int_0^t \langle u \otimes \varpi(f), k_s\big(E^{\widehat{f(s)}} \psi(x) E_{\widehat{g(s)}}\big) v \otimes \varpi(g)\rangle \, \mathrm{d}s$$

for all $u,\ v \in \mathsf{h},\ f,\ g \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ and $t \in \mathbb{R}_+$. If this holds for all $x \in \mathsf{V}_0$ then k is a *weak solution* of the Evans–Hudson QSDE in the terminology of [16, 18, 19]. A *strong solution* must, in addition, satisfy the extra requirement that $\big(\widetilde{k}_t(\psi(x))\big)_{t \in \mathbb{R}_+}$ be integrable, as explained below in Definition 4.11.

Remark 3.13. Recall that a C_0 *semigroup* on the Banach space X is a family $(T_t)_{t \in \mathbb{R}_+} \subseteq B(X)$ such that

$$T_0 = I_X, \qquad T_{s+t} = T_s \circ T_s \quad \text{for all } s, t \in \mathbb{R}_+ \quad \text{and} \quad \lim_{t \to 0} \|T_t(x) - x\| = 0 \quad \text{for all } x \in X.$$

It follows from these assumptions that the map

$$\mathbb{R}_+ \times X \to X;\ (t, x) \mapsto T_t(x)$$

is jointly continuous [12, Theorem 6.2.1].

Any C_0 semigroup is characterised by its *infinitesimal generator* $\tau : \mathrm{dom}\, \tau \to X$, the closed, densely defined operator such that

$$\mathrm{dom}\, \tau = \big\{x \in X : \lim_{t \to 0} t^{-1}(T_t(x) - x) \text{ exists}\big\} \quad \text{and} \quad \tau(x) = \lim_{t \to 0} t^{-1}(T_t(x) - x).$$

Note that $\operatorname{dom}\tau$ is left invariant by T_t [12, Lemma 6.1.11]. A *core* for τ is a subspace X_0 of $\operatorname{dom}\tau$ such that the graph of the restriction $\tau|_{X_0}$ is dense in the graph of τ, regarded as a subset of the normed space $X \oplus X$ with the subspace topology.

Notation 3.14. Let $\chi : \mathsf{k} \times \mathsf{k} \to \mathbb{C}$ denote the map $\chi(z, w) = \frac{1}{2}\|z\|^2 + \frac{1}{2}\|w\|^2 - \langle z, w\rangle$.

Theorem 3.15. *Let k be a completely bounded mapping process on V with locally bounded norm. Suppose there exists an admissible set T, a norm-dense subspace V_0 of V and a linear map $\psi : \mathsf{V}_0 \to \mathsf{V} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ such that*

$$k_0(x) = x \otimes I \qquad \text{and} \qquad \mathrm{d}k_t(x) = (\widetilde{k}_t \circ \psi)(x)\,\mathrm{d}\Lambda_t \quad \text{weakly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T})$$

for all $x \in \mathsf{V}_0$. If, for all z, $w \in \mathsf{T}$, there exists a C_0 semigroup generator $\eta_{z,w} : \operatorname{dom}\eta_{z,w} \to \mathsf{V}$ with a core $\mathsf{V}_{z,w} \subseteq \mathsf{V}_0$ such that $\eta_{z,w}(x) = E^{\widehat{z}}\psi(x)E_{\widehat{w}}$ for all $x \in \mathsf{V}_{z,w}$ then k is a Markovian cocycle.

Proof. If f, $g \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ are constant on $[s, t) \subseteq \mathbb{R}_+$, say $f(s) = z$ and $g(s) = w$, then

$$\langle u \otimes \varpi(f), \big(k_t(x) - k_s(x)\big)v \otimes \varpi(g)\rangle = \int_s^t \langle u \otimes \varpi(f), k_r\big(\eta_{z,w}(x)\big)v \otimes \varpi(g)\rangle \,\mathrm{d}r$$

for all u, $v \in \mathsf{h}$ and $x \in \mathsf{V}_{z,w}$. Given $y \in \operatorname{dom}\eta_{z,w}$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{V}_{z,w}$ such that

$$\|x_n - y\| + \|\eta_{z,w}(x_n - y)\| \to 0 \qquad \text{as } n \to \infty$$

and therefore Lebesgue's Dominated Convergence Theorem implies that, in the weak sense,

$$k_t(f, g)(y) - k_s(f, g)(y) = \int_s^t k_r(f, g)\big(\eta_{z,w}(y)\big)\,\mathrm{d}r \qquad \text{for all } y \in \operatorname{dom}\eta_{z,w},$$

where $k_r(f, g)(y) := E^{\varpi(f)}k_r(y)E_{\varpi(g)}$ and so on. In particular, the function $r \mapsto k_r(f, g)(y)$ is weakly continuous on $[s, t]$ for all $y \in \operatorname{dom}\eta_{z,w}$, so for all $y \in \mathsf{V}$.

We now follow [18, Proof of Proposition 3.4]. Let f, $g \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ be subordinate to the partition $\{0 = t_0 < t_1 < \cdots\}$, let $t \in (t_n, t_{n+1}]$ for some $n \in \mathbb{Z}_+$, let $y \in \operatorname{dom}\eta_{z,w}$, let u, $v \in \mathsf{h}$ and consider the function

$$F : [t_n, t] \to \mathbb{C}; \; s \mapsto \langle u, k_s(f, g)\big(\mathcal{Q}^{z,w}_{t-s}(y)\big)v\rangle,$$

where $(\mathcal{Q}^{z,w}_t)_{t\in\mathbb{R}_+}$ is the C_0 semigroup on V generated by $\eta_{z,w}$, with $z = f(t_n)$ and $w = g(t_n)$. The weak continuity of $r \mapsto k_r(f, g)(x)$ for all $x \in \mathsf{V}$ and the norm continuity of $r \mapsto \mathcal{Q}^{z,w}_r(y)$, together with the locally uniform boundedness of the norm of k, give the continuity of F.

If $s \in [t_n, t)$ and $h \in (0, t - s)$ then, since $\operatorname{dom}\eta_{z,w}$ is invariant under $\mathcal{Q}^{z,w}$,

$$\begin{aligned} h^{-1}\big(F(s + h) - F(s)\big) &= \langle u, (k_{s+h} - k_s)(f, g)\big(h^{-1}(\mathcal{Q}^{z,w}_{t-s-h} - \mathcal{Q}^{z,w}_{t-s})(y)\big)v\rangle \\ &\quad + h^{-1}\langle u, (k_{s+h} - k_s)(f, g)\big(\mathcal{Q}^{z,w}_{t-s}(y)\big)v\rangle \\ &\quad + \langle u, k_s(f, g)\big(h^{-1}\big(\mathcal{Q}^{z,w}_{t-s-h} - \mathcal{Q}^{z,w}_{t-s}\big)(y)\big)v\rangle \\ &\to 0 + \langle u, k_s(f, g)\big(\eta_{z,w}(\mathcal{Q}^{z,w}_{t-s}(y))\big)v\rangle - \langle u, k_s(f, g)\big(\eta_{z,w}(\mathcal{Q}^{z,w}_{t-s}(y))\big)v\rangle = 0 \end{aligned}$$

as $h \to 0$, so F is constant on $[t_n, t]$, being a continuous function with vanishing right derivative. Hence

$$k_t(f,g)(y) = \big(k_{t_n}(f,g) \circ \mathcal{Q}^{z,w}_{t-t_n}\big)(y) \qquad \text{for all } y \in \operatorname{dom} \eta_{z,w},$$

and so for all $y \in \mathsf{V}$. Repeating this argument gives that

$$k_t(f,g) = \langle \varpi(f), \varpi(g) \rangle \, \mathcal{Q}^{z_0,w_0}_{t_1-t_0} \circ \cdots \circ \mathcal{Q}^{z_n,w_n}_{t-t_n}, \tag{3.2}$$

where now $f(t_j) = z_j$ and $g(t_j) = w_j$ for $j = 0, \ldots, n$. In particular, for any $z, w \in \mathsf{T}$,

$$\mathcal{P}^{z,w}_s = k_s[1_{[0,s)}z, 1_{[0,s)}w] = k_s(1_{[0,s)}z, 1_{[0,s)}w) = \exp(-s\chi(z,w))\mathcal{Q}^{z,w}_s.$$

This relationship between $\mathcal{P}^{z,w}$ and $\mathcal{Q}^{z,w}$, together with (3.2), yields

$$k_t[f,g] = \mathcal{P}^{z_0,w_0}_{t_1-t_0} \circ \cdots \circ \mathcal{P}^{z_n,w_n}_{t-t_n},$$

and therefore k is a Markovian cocycle, by Theorem 3.10. □

Proposition 3.16. *Let k be a Markovian cocycle on V with locally bounded norm. Suppose there exists an admissible set T, a norm-dense subspace V_0 of V and a linear map $\psi : \mathsf{V}_0 \to \mathsf{V} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ such that*

$$k_0(x) = x \otimes I \qquad \textit{and} \qquad \mathrm{d}k_t(x) = (\widetilde{k}_t \circ \psi)(x)\, \mathrm{d}\Lambda_t \quad \textit{weakly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \tag{3.3}$$

for all $x \in \mathsf{V}_0$. If one of the associated semigroups $\mathcal{P}^{z,w}$ of the cocycle is C_0 for some choice of $z, w \in \mathsf{k}$ then all of them are C_0 semigroups. Moreover $\psi^{\widehat{z}}_{\widehat{w}} := E^{\widehat{z}} \psi(\cdot) E_{\widehat{w}} : \mathsf{V}_0 \to \mathsf{V}$ is closable for all $z, w \in \mathsf{T}$ and the generator of $\mathcal{P}^{z,w}$ is an extension of the map $x \mapsto \psi^{\widehat{z}}_{\widehat{w}}(x) - \chi(z,w)x$.

Proof. Let $t > 0$ and $x \in \mathsf{V}_0$. Since k satisfies the QSDE (3.3) weakly, it follows that

$$t^{-1}\big(\mathcal{P}^{z,w}_t(x) - x\big) = t^{-1}\left(\int_0^t e^{(s-t)\chi(z,w)} \mathcal{P}^{z,w}_s\big(\psi^{\widehat{z}}_{\widehat{w}}(x)\big)\, \mathrm{d}s + (e^{-t\chi(z,w)} - 1)x\right) \tag{3.4}$$

in the weak operator sense. However it is known [20, Proposition 5.4] that either all or none of the associated semigroups are C_0; if they do all have this property then the integrand is a norm-continuous map from $\mathbb{R}_+$ to V, so equation (3.4) in fact makes sense as a Bochner integral. Note then that the limit of the right-hand side exists as $t \to 0$, showing that V_0 is contained in the domain of the generator of $\mathcal{P}^{z,w}$, whose action on V_0 is as claimed. □

4 Feynman–Kac perturbation

4.1 The free flow

Definition 4.1. A *quantum stochastic flow* is a Markovian cocycle j on a unital C^* algebra $\mathsf{A} \subseteq B(\mathsf{h})$ such that each j_t is a unital $*$-homomorphism and the mapping process j is strongly continuous.

Remark 4.2. Note that in this paper we insist that a flow be a cocycle but make no requirement that it solve a QSDE. This is at somewhat at odds with the terminology adopted elsewhere.

Definition 4.3. Let j be a quantum stochastic flow on the unital C^* algebra A and let T be an admissible set. Suppose that

$$\mathrm{d}j_t(x) = (\widetilde{\jmath}_t \circ \phi)(x)\,\mathrm{d}\Lambda_t \qquad \text{weakly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \qquad \text{for all } x \in \mathsf{A}_0, \tag{4.1}$$

where A_0 is a subspace of A and $\phi : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$. The map ϕ, called the *generator* of the flow j, has *standard form* if A_0 is a norm-dense $*$-algebra containing the multiplicative unit 1_{A} and ϕ is a linear map with the block-matrix decomposition

$$\phi = \begin{bmatrix} \tau & \delta^\dagger \\ \delta & \pi - \iota \end{bmatrix} : \mathsf{A}_0 \to \begin{bmatrix} \mathsf{A} & \mathsf{A} \otimes_{\mathrm{m}} \langle \mathsf{k}| \\ \mathsf{A} \otimes_{\mathrm{m}} |\mathsf{k}\rangle & \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k}) \end{bmatrix}, \tag{4.2}$$

where

(i) $\tau : \mathsf{A}_0 \to \mathsf{A}$ is a $*$-linear map such that

$$\tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \qquad \text{for all } x, y \in \mathsf{A}_0,$$

(ii) $\delta : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} |\mathsf{k}\rangle$ is a π-derivation, so that

$$\delta(xy) = \delta(x)y + \pi(x)\delta(y) \qquad \text{for all } x, y \in \mathsf{A}_0,$$

(iii) $\delta^\dagger : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} \langle \mathsf{k}|$ is such that $\delta^\dagger(x) = \delta(x^*)^*$ for all $x \in \mathsf{A}_0$, and

(iv) $\pi : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k})$ is a unital $*$-homomorphism and $\iota : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k})$ is the ampliation map $x \mapsto x \otimes I_{\mathsf{k}}$.

Remark 4.4. Suppose (4.1) holds with $\mathsf{A}_0 = \mathsf{A}$.

(i) If the multiplicity space k is finite dimensional and ϕ is bounded, then j satisfies (4.1) strongly and ϕ has standard form [18, Theorem 5.1(d)].

(ii) More generally, if there exists an orthonormal basis $\{e_\alpha : \alpha \in A\}$ of $\widehat{\mathsf{k}}$ that contains the vector $(1, 0)$ and is such that

$$\mathsf{A} \to B(\mathsf{h});\ x \mapsto E^{e_\alpha}\phi(x)E_{e_\beta}$$

is a bounded map for all α, $\beta \in A$ then ϕ is completely bounded [19, Proposition 5.1 and Theorem 5.2] and has standard form [19, Proposition 3.2, Proposition 6.3 and Theorem 6.5].

Example 4.5. Suppose A is a unital C^* algebra, let $t \in \mathsf{A} \otimes_{\mathrm{m}} |\mathsf{k}\rangle$ be such that $t^*(x \otimes I_{\mathsf{k}})t \in \mathsf{A}$ for all $x \in \mathsf{A}$ and let $h = h^* \in \mathsf{A}$. The map

$$\phi = \begin{bmatrix} \tau & \delta^\dagger \\ \delta & 0 \end{bmatrix} \tag{4.3}$$

has standard form, where

$$\delta(x) = (x \otimes I_{\mathsf{k}})t - tx$$
$$\text{and} \quad \tau(x) = \mathrm{i}[h, x] - \tfrac{1}{2}t^*tx + t^*(x \otimes I_{\mathsf{k}})t - \tfrac{1}{2}xt^*t \qquad \text{for all } x \in \mathsf{A}.$$

Example 4.6. Let A be a unital C^* algebra with a norm-dense $*$-subalgebra $\mathsf{A}_0 \subseteq \mathsf{A}$ containing the multiplicative unit 1_{A}. For $i = 1, \ldots, n$, let $c_i \in \mathbb{C}$ and let $\delta_i : \mathsf{A}_0 \to \mathsf{A}$ be a *skew-symmetric* derivation, so that $\delta_i^\dagger = -\delta_i$. With $\{e_1, \ldots, e_n\}$ the standard orthonormal basis of $\mathsf{k} = \mathbb{C}^n$,

$$\delta(x) := \sum_{i=1}^n c_i \delta_i(x) \otimes |e_i\rangle \qquad \text{and} \qquad \tau(x) := -\frac{1}{2}\sum_{i=1}^n |c_i|^2 \delta_i^2(x) \qquad \text{for all } x \in \mathsf{A}_0,$$

the map ϕ given by (4.3) has standard form.

Remark 4.7. The challenge is to extend these paradigmatic examples. The map ϕ defined in Example 4.5 is completely bounded, so the QSDE (4.1) can be solved [21] and the solution is a strong one. It is natural to ask what happens when τ is no longer bounded, for instance when the derivations $\delta_1, \ldots, \delta_n$ in Example 4.6 are unbounded. Can we still solve (4.1)? These challenges are addressed in Section 5.

4.2 The multiplier equation

Definition 4.8. A bounded operator process X is a *right multiplier cocycle* for the quantum stochastic flow j [6, Definition 2.2] if

$$X_{s+t} = J_s(X_t)X_s \qquad \text{for all } s, t \in \mathbb{R}_+,$$

where $J_s := \widehat{\jmath}_s \circ \sigma_s$ as in Proposition 3.8.

Proposition 4.9. *Let j be a quantum stochastic flow on the unital C^* algebra A. Suppose k is a Markovian cocycle on A and Y is a bounded operator process such that*

$$k_t(x) = j_t(x)Y_t \qquad \text{for all } t \in \mathbb{R}_+ \text{ and } x \in \mathsf{A}.$$

Then Y is a right multiplier cocycle for the flow j.

Proof. Note first that

$$k_{t)}(x) = j_{t)}(x)Y_{t)} \quad \text{and} \quad Y_t E_{\varpi(g)} = E_{\varpi(g_{[t})} Y_{t)} E_{\varpi(g_{t)})} \qquad \text{for all } t \in \mathbb{R}_+ \text{and} x \in \mathsf{A},$$

using the notation of Definitions 3.3 and 3.5. Consequently we have that $\widehat{k}_t(R) = \widehat{\jmath}_t(R)Y_t$ for all $R \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathcal{F}_{[t})$, thus $K_t(T) = J_t(T)Y_t$ for all $t \in \mathbb{R}_+$ and $T \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathcal{F})$ in the notation of Remark 3.7 and Proposition 3.8. Hence

$$Y_{s+t} = k_{s+t}(I_{\mathsf{h}}) = (\widehat{k}_s \circ \sigma_s \circ k_t)(I_{\mathsf{h}}) = K_s(Y_t) = J_s(Y_t)Y_s \qquad \text{for all } s, t \in \mathbb{R}_+. \qquad \square$$

Remark 4.10. Proposition 4.9 reverses the reasoning used in the von Neumann-algebraic context, where it is shown that k is a Markovian cocycle if Y is a suitable right multiplier cocycle [6, Proposition 2.5]. The obstruction in the C^* setting appears when attempting to show that $j_t(a)Y_t$ lies in $\mathsf{A} \otimes_{\mathrm{m}} B(\mathcal{F})$: see Remark 4.18.

Definition 4.11. Given an admissible set T, an *integrable process* F is a collection of linear operators $(F_t)_{t\in\mathbb{R}_+}$ such that

(i) $\mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \underline{\otimes} \widehat{\mathsf{k}} \subseteq \operatorname{dom} F_t \subseteq \mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$ and $\operatorname{im} F_t \subseteq \mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$ for all $t \in \mathbb{R}_+$,

(ii) $E^{\varpi(f)\otimes\xi} F_t E_{\varpi(g)\otimes\eta} = \langle \varpi(1_{[t,\infty)}f), \varpi(1_{[t,\infty)}g)\rangle E^{\varpi(1_{[0,t)}f)\otimes\xi} F_t E_{\varpi(1_{[0,t)}g)\otimes\eta}$ for almost all $t \in \mathbb{R}_+$, for all f, $g \in L^{\mathrm{step}}(\mathbb{R}_+;\mathsf{T})$ and ξ, $\eta \in \widehat{\mathsf{k}}$,

(iii) $t \mapsto F_t\xi$ is strongly measurable for all $\xi \in \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \underline{\otimes} \widehat{\mathsf{k}}$,

(iv) $t \mapsto \Delta^{\perp}_{\mathsf{h}\otimes\mathcal{F}} F_t\big(u \otimes \varpi(f) \otimes \widehat{f(t)}\big)$ is locally integrable for all $u \in \mathsf{h}$ and $f \in L^{\mathrm{step}}(\mathbb{R}_+;\mathsf{T})$, and

(v) $t \mapsto \Delta_{\mathsf{h}\otimes\mathcal{F}} F_t\big(u \otimes \varpi(f) \otimes \widehat{f(t)}\big)$ is locally square-integrable for all $u \in \mathsf{h}$ and $f \in L^{\mathrm{step}}(\mathbb{R}_+;\mathsf{T})$.

The integrable process F is *bounded* if $F_t \in B(\mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}})$ for all $t \in \mathbb{R}_+$.

Given such an integrable process F, the quantum stochastic integral $(\int_0^t F_s \,\mathrm{d}\Lambda_s)_{t\in\mathbb{R}_+}$ is the unique operator process such that

$$\langle u \otimes \varpi(f), \int_0^t F_s \,\mathrm{d}\Lambda_s v \otimes \varpi(g)\rangle = \int_0^t \langle u \otimes \varpi(f), E^{\widehat{f(s)}} F_s E_{\widehat{g(s)}} v \otimes \varpi(g)\rangle \,\mathrm{d}s$$

for all $t \in \mathbb{R}_+$, u, $v \in \mathsf{h}$ and f, $g \in L^{\mathrm{step}}(\mathbb{R}_+;\mathsf{T})$. Such a process is necessarily strongly continuous. For details see Theorem 3.13 of [16].

Remark 4.12. (i) Definition 4.11(iii) implies that a bounded integrable process F is strongly measurable everywhere on $\mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$, that is, the map $t \mapsto F_t\theta$ is strongly measurable for all $\theta \in \mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$.

(ii) If X is a strongly measurable bounded operator process on h which has locally bounded norm then $\widetilde{X} := (X_t \otimes I_{\widehat{\mathsf{k}}})_{t\in\mathbb{R}_+}$ is a bounded integrable process with locally bounded norm.

Lemma 4.13. *Let $R \in B(\mathsf{h})$ and let L be a bounded integrable process with locally bounded norm. There is a unique strong solution to the QSDE*

$$X_0 = R \otimes I_{\mathcal{F}} \qquad \text{and} \qquad \mathrm{d}X_t = L_t \widetilde{X}_t \,\mathrm{d}\Lambda_t, \tag{4.4}$$

where $\widetilde{X_t} := X_t \underline{\otimes} I_{\widehat{\mathsf{k}}}$. The solution X is a strongly continuous operator process and $\widetilde{X}$ is an integrable process such that $t \mapsto X_t E_{\varpi(f)}$ has locally bounded norm for every $f \in L^{\mathrm{step}}(\mathbb{R}_+;\mathsf{T})$.

Proof. This is essentially Proposition 3.1 of [15], rewritten in the notation and language of [16]. In particular X is found by solving the iteration scheme

$$X_t^{(0)} = R \otimes I_{\mathcal{F}}, \qquad X_t^{(n+1)} = R \otimes I_{\mathcal{F}} + \int_0^t L_s \widetilde{X}_s^{(n)} \, \mathrm{d}\Lambda_s, \quad t \in \mathbb{R}_+, n \in \mathbb{Z}_+$$

where $\widetilde{X}_t^{(n)} := X_t^{(n)} \underline{\otimes} I_{\widehat{\mathsf{k}}}$. The norm continuity of $t \mapsto \widetilde{X}_t^{(n)}\xi$ for all $\xi \in \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \underline{\otimes} \widehat{\mathsf{k}}$ implies that $s \mapsto L_s \widetilde{X}_s^{(n)}$ is an integrable process, giving the existence of $X^{(n+1)}$. We fix $f \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ and let

$$Y_t^{(0)} := (R \otimes I_{\mathcal{F}}) E_{\varpi(f)} \qquad \text{and} \qquad Y_t^{(n)} := (X_t^{(n+1)} - X_t^{(n)}) E_{\varpi(f)}.$$

Given $T > 0$ and applying [16, Theorem 3.13], it follows by induction that $Y_t^{(n)} \in B(\mathsf{h}; \mathsf{h} \otimes \mathcal{F})$ for all $n \in \mathbb{N}$, with

$$\sup_{t \in [0,T]} \|Y_t^{(n)} u\|^2 \leqslant K_{f,T} \int_0^t \|L_s\|^2 \|\widehat{f}(s)\|^2 \|Y_s^{(n-1)} u\|^2 \, \mathrm{d}s,$$

where $K_{f,T}$ is a constant that depends only on f and T. Consequently, we have that

$$\sup_{t \in [0,T]} \|Y_t^{(n)} u\| \leqslant \frac{1}{\sqrt{n!}} K_{f,T}^{n/2} M_T^n \|1_{[0,T]} \widehat{f}\|^n \|R\| \|\varpi(f)\| \|u\|,$$

where $M_T = \sup_{t \in [0,T]} \|L_t\|$. Setting $X_t u \otimes \varpi(f) = \sum_{n=0}^\infty Y_t^{(n)} u$ for all $u \in \mathsf{h}$ and $t \in \mathbb{R}_+$ gives a process X. It is now a routine matter to check that X is strongly continuous, and integrability of $\widetilde{X}$ then follows, in part because of the manifestly locally bounded norm of $t \mapsto X_t E_{\varpi(f)}$. Thus X can be shown to satisfy (4.4). Uniqueness follows by taking the difference of two solutions and iterating as above. □

Proposition 4.14. *Let j be a completely bounded mapping process on the operator space V, with locally bounded completely bounded norm. If $F \in \mathsf{V} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ is such that $t \mapsto \widetilde{\jmath}_t(F)$ is strongly measurable then there is a unique strong solution to the QSDE*

$$X_0 = I_{\mathsf{h} \otimes \mathcal{F}} \qquad \textit{and} \qquad \mathrm{d}X_t = \widetilde{\jmath}_t(F) \widetilde{X}_t \, \mathrm{d}\Lambda_t. \tag{4.5}$$

The solution X is a strongly continuous process. If $\mathsf{V} = \mathsf{A}$, a unital C^* algebra, with each j_t being a $*$-homomorphism and such that

$$\widetilde{\jmath}_t(F)^* \Delta_{\mathsf{h} \otimes \mathcal{F}} \widetilde{\jmath}_t(F) = \widetilde{\jmath}_t(F^* \Delta_{\mathsf{h}} F), \tag{4.6}$$

then X is contractive if and only if

$$q(F) := F + F^* + F^* \Delta_{\mathsf{h}} F \leqslant 0$$

and X is isometric if and only if $q(F) = 0$.

Proof. The existence of X is an application of Lemma 4.13. The extra hypotheses on V, j and F, together with the weak form of the quantum Itô product formula [16, Theorem 3.15], imply that

$$\|X_t\theta\|^2 - \|\theta\|^2 = \int_0^t \langle \widetilde{X}_s \widehat{\nabla}_s \theta, \widetilde{\jmath}_s(F + F^* + F^*\Delta F)\widetilde{X}_s\widehat{\nabla}_s\theta\rangle \,\mathrm{d}s \quad \text{for all } \theta \in \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \text{ and } t \in \mathbb{R}_+,$$

where $\widehat{\nabla}_s u \otimes \varpi(f) := u \otimes \varpi(f) \otimes \widehat{f(s)}$ and this definition is extended by linearity. Sufficiency of the isometry and contractivity conditions follows immediately; for the latter, recall that matrix-space liftings preserve completely positivity, by Lemma 2.4. Necessity follows by differentiating at 0: the integrand is continuous at 0 by Remark 3.4(ii). □

Remark 4.15. (i) If $\mathsf{V} = \mathsf{A}$, a unital C^* algebra, and j_t is a $*$-homomorphism, then (4.6) holds whenever $F \in \mathsf{A} \otimes B(\widehat{\mathsf{k}})$.

(ii) Suppose that $I_\mathsf{h} \in \mathsf{V}$ and $j_t(I_\mathsf{h}) = I_{\mathsf{h}\otimes\mathcal{F}}$ for all $t \in \mathbb{R}_+$. If $C \in B(\widehat{\mathsf{k}})$ and $F = I_\mathsf{h} \otimes C$ then the conditions of Proposition 4.14 are satisfied and the strongly continuous operator process X such that

$$X_t = I_{\mathsf{h}\otimes\mathcal{F}} + \int_0^t X_s \underline{\otimes}\, C \,\mathrm{d}\Lambda_s \qquad \text{strongly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \qquad \text{for all } t \in \mathbb{R}_+$$

is contractive, isometric or co-isometric if and only if $q(F) \leqslant 0$, $q(F) = 0$ or $q(F^*) = 0$, respectively. This follows from the quantum Itô product formula, Theorem 4.24; alternatively, note that this is the usual Hudson–Parthasarathy QSDE with time-independent coefficients.

Lemma 4.16. *Let j be a completely bounded mapping process on the operator space V, with locally bounded completely bounded norm.*

(i) *The process $t \mapsto \widetilde{\jmath}_t(F)$ is weakly measurable for all $F \in \mathsf{V} \otimes_\mathrm{m} B(\widehat{\mathsf{k}})$ if and only if the process $t \mapsto j_t(x)$ is weakly measurable for all $x \in \mathsf{A}$.*

(ii) *If $t \mapsto j_t(x)$ is strongly measurable for all $x \in \mathsf{V}$ then $t \mapsto \widetilde{\jmath}_t(F)$ is strongly measurable for all $F \in \mathsf{A} \otimes B(\widehat{\mathsf{k}})$, the spatial tensor product.*

(iii) *If j is strongly continuous then $t \mapsto \widetilde{\jmath}_t(F)$ is strongly measurable for all $F \in \mathsf{V} \otimes_\mathrm{m} B(\widehat{\mathsf{k}})$, the matrix-space tensor product.*

Proof. Parts (i) and (ii) are easily checked.

For part (iii), it is enough to show that if $F \in \mathsf{V} \otimes_\mathrm{m} B(\widehat{\mathsf{k}})$, $u \in \mathsf{h}$, $f \in L^{\mathrm{step}}(\mathbb{R}_+; \mathsf{T})$ and $z \in \widehat{\mathsf{k}}$ then the map $t \mapsto \widetilde{\jmath}_t(F)\zeta$ is strongly measurable, where $\zeta := u \otimes \varpi(f) \otimes z$.

We fix an orthonormal basis $\{e_\alpha\}_{\alpha\in A}$ of $\widehat{\mathsf{k}}$ and let $E^\alpha := E^{e_\alpha}$. Given any $\xi \in \mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$, we have that $\|\xi\|^2 = \sum_{\alpha\in A} \|E^\alpha \xi\|^2$ and so $E^\alpha\xi \neq 0$ for only countably many α. Thus there is a countable set $A_t \subseteq A$ for each $t \in \mathbb{R}_+$ such that $E^\alpha \widetilde{\jmath}_t(F)\zeta = 0$ when $\alpha \notin A_t$. We let $\mathsf{K}_\mathbb{Q}$ be the separable closed subspace of $\widehat{\mathsf{k}}$ with orthonormal basis $\{e_\alpha : \alpha \in A_\mathbb{Q}\}$, where $A_\mathbb{Q} := \bigcup\{A_t : t \in \mathbb{Q} \cap \mathbb{R}_+\}$.

Next, for each $\alpha \in A_{\mathbb{Q}}$ we know that $t \mapsto j_t(E^\alpha F E_z)u \otimes \varpi(f)$ is continuous, so its image is contained in a separable subspace H_α of $\mathsf{h} \otimes \mathcal{F}$. Let $\mathsf{H}_{\mathbb{Q}}$ denote the smallest closed subspace of $\mathsf{h} \otimes \mathcal{F}$ that contains every H_α, which will also be separable. Then for each $t \in \mathbb{Q} \cap \mathbb{R}_+$ we have that

$$\widetilde{\jmath}_t(F)\zeta = \sum_{\alpha \in A_t} \big[j_t(E^\alpha F E_z)u \otimes \varpi(f)\big] \otimes 3e_\alpha \in \mathsf{H}_{\mathbb{Q}} \otimes \mathsf{K}_{\mathbb{Q}}.$$

Finally, for each $t \in \mathbb{R}_+ \setminus \mathbb{Q}$, we have that

$$\langle \nu, \widetilde{\jmath}_t(F)\zeta \rangle = \lim_{n \to \infty} \langle \nu, \widetilde{\jmath}_{t_n}(F)\zeta \rangle$$

for any $\nu \in \mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$ and any sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{Q} \cap \mathbb{R}_+$ that converges to t, by the strong, and hence weak, continuity of j. Thus $\widetilde{\jmath}_t(F)\zeta$ belongs to the weak closure of $\mathsf{H}_{\mathbb{Q}} \otimes \mathsf{K}_{\mathbb{Q}}$, which coincides with the norm closure (as these closures coincide for convex subsets of any normed vector space).

This shows that the vector process $(\widetilde{\jmath}_t(F)\zeta)_{t \in \mathbb{R}_+}$ is separably valued. The result now follows from Pettis' Theorem, given that this process is also weakly measurable as a result of the local boundedness of the completely bounded norm of j and its strong continuity. □

Proposition 4.17. *Let* A *be a unital* C^* *algebra with positive cone* A_+ *and suppose*

$$F = \begin{bmatrix} k & m \\ l & w - I_{\mathsf{h} \otimes \mathsf{k}} \end{bmatrix} \in \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}}).$$

Then $q(F) \leqslant 0$ *if and only if* $w \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k})$ *is a contraction,* $d := -(k + k^* + l^* l) \in \mathsf{A}_+$ *and there exists a contraction* v *such that* $m = -l^* w - d^{1/2} v (I_{\mathsf{h} \otimes \mathsf{k}} - w^* w)^{1/2}$.

Furthermore, $q(F) = 0$ *if and only if* $w^* w = I_{\mathsf{h} \otimes \mathsf{k}}$, $k + k^* + l^* l = 0$ *and* $m = -l^* w$.

Proof. The first part follows from standard characterisations of positive 2×2 operator matrices (see [14, Lemma 2.1]). The second part is immediate. □

4.3 Perturbation of the free flow

Remark 4.18. If $\mathsf{A} \subseteq B(\mathsf{h})$ is a unital C^* algebra and H is a Hilbert space then, in general, the operator space $\mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{H})$ is not an algebra, although it is always an operator system in $B(\mathsf{h} \otimes \mathsf{H})$: see [21, pp. 615–6]. The reason is already apparent at the level of row and column spaces, since $\mathsf{A} \otimes |\mathsf{H}\rangle$ and $\mathsf{A} \otimes_{\mathrm{m}} |\mathsf{H}\rangle$ typically differ, with the former having a natural Hilbert C^*-bimodule structure not shared with the latter. Moreover, if $\{e_\alpha\}_{\alpha \in A}$ is any orthonormal basis of H then for each $T \in \mathsf{A} \otimes |\mathsf{H}\rangle$ it is the case that

$$T = \sum_{\alpha \in A} E^{e_\alpha} T \otimes |e_\alpha\rangle = \sum_{\alpha \in A} E_{e_\alpha} E^{e_\alpha} T$$

with the series being norm convergent, whereas if $T \in \mathsf{A} \otimes_{\mathrm{m}} |\mathsf{H}\rangle$ then the series above converges to T in the strong operator topology, but not necessarily in norm.

The following construction provides a means of avoiding some of the problems caused by this inconvenient difference.

Definition 4.19. Let $\mathsf{V} \subseteq B(\mathsf{h})$ be an operator space, and let H be a Hilbert space. Let

$$R(\mathsf{V};\mathsf{H}) := \{T \in B(\mathsf{h} \otimes \mathsf{H}) : E^z T \in \mathsf{V} \otimes \langle \mathsf{H}| \text{ for all } z \in \mathsf{H}\}$$
$$\text{and} \quad C(\mathsf{V};\mathsf{H}) := \{T \in B(\mathsf{h} \otimes \mathsf{H}) : T E_w \in \mathsf{V} \otimes |\mathsf{H}\rangle \text{ for all } w \in \mathsf{H}\}.$$

Lemma 4.20. *Let T_R and T_C denote the topologies on $B(\mathsf{h} \otimes \mathsf{H})$ generated by the families of seminorms $\{{}^z p\}_{z \in \mathsf{H}}$ and $\{p_w\}_{w \in \mathsf{H}}$, respectively, where*

$${}^z p(T) := \|E^z T\| \quad \textit{and} \quad p_w(T) := \|T E_w\| \qquad \textit{for all } T \in B(\mathsf{h} \otimes \mathsf{H}).$$

Then $R(\mathsf{V};\mathsf{H}) = \overline{\mathsf{V} \underline{\otimes} B(\mathsf{H})}^R$ and $C(\mathsf{V};\mathsf{H}) = \overline{\mathsf{V} \underline{\otimes} B(\mathsf{H})}^C$, the closures with respect to T_R and T_C, respectively. Moreover

$$\mathsf{V} \otimes B(\mathsf{H}) \subseteq R(\mathsf{V};\mathsf{H}) \cap C(\mathsf{V};\mathsf{H}) \subseteq R(\mathsf{V};\mathsf{H}) \cup C(\mathsf{V};\mathsf{H}) \subseteq \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H}).$$

Proof. Any $T \in \overline{\mathsf{V} \underline{\otimes} B(\mathsf{H})}^R$ is the limit of a T_R-convergent net $\{T_i\}_{i \in I} \subseteq \mathsf{V} \underline{\otimes} B(\mathsf{H})$. We have that $E^z T_i \in \mathsf{V} \underline{\otimes} \langle \mathsf{H}| \subseteq \mathsf{V} \otimes \langle \mathsf{H}|$, with the latter space being norm closed. Thus $T \in R(\mathsf{V};\mathsf{H})$.

Next, let $S \in R(\mathsf{V};\mathsf{H})$. Given an orthonormal basis $\{e_\alpha\}_{\alpha \in A}$ of H, for each finite set $B \subset\subset A$ we let

$$p_B := \sum_{\beta \in B} |e_\beta\rangle\langle e_\beta| \quad \text{and} \quad P_B := I_{\mathsf{h}} \otimes p_B = \sum_{\beta \in B} E_{e_\beta} E^{e_\beta}.$$

For any finite set $C \subset\subset A$, Remark 4.18 with $T = (P_C S)^*$ gives that

$$\lim_{B \subset\subset A} \|P_C S P_B - P_C S\| = 0.$$

We now consider the net $\{P_B S P_B\}_{B \subset\subset A} \subseteq \mathsf{V} \underline{\otimes} B(\mathsf{H})$. Given any $z \in \mathsf{H}$ and $\varepsilon > 0$, there is a finite set $B_0 \subset\subset A$ such that $\|z - p_{B_0} z\| < \varepsilon$. Then $P_B P_{B_0} = P_{B_0}$ for any finite set $B \supseteq B_0$ and so

$${}^z p(P_B S P_B - S) \leqslant 2\varepsilon \|S\| + \|z\| \, \|P_{B_0} S P_B - P_{B_0} S\|.$$

Hence $R(\mathsf{V};\mathsf{H}) \subseteq \overline{\mathsf{V} \underline{\otimes} B(\mathsf{H})}^R$, as required. That $C(\mathsf{V};\mathsf{H}) = \overline{\mathsf{V} \underline{\otimes} B(\mathsf{H})}^C$ is proved in the same way. The inclusions are readily verified, in particular as it is clear that T_R and T_C are weaker than the norm topology. □

Lemma 4.21. *Let $\phi : \mathsf{A} \to \mathsf{B}$ be a $*$-homomorphism between the C^* algebras A and B and suppose $S, T \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{H})$, where H is a Hilbert space. If either $S \in R(\mathsf{A};\mathsf{H})$ or $T \in C(\mathsf{A};\mathsf{H})$ then $ST \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{H})$ and*

$$(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(S)(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(T) = (\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(ST).$$

Proof. For any orthonormal basis $\{e_\alpha\}_{\alpha\in A}$ of H and any z, $w \in \mathsf{H}$ we have that

$$E^z STE_w = \sum_{\alpha\in A} E^z SE_{e_\alpha} E^{e_\alpha} TE_w \in \mathsf{A},$$

as this series is norm convergent by Remark 4.18. Hence

$$\begin{aligned} E^z(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(S)(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(T)E_w &= \sum_{\alpha\in A} E^z(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(S)E_{e_\alpha} E^{e_\alpha}(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(T)E_w \\ &= \sum_{\alpha\in A} \phi(E^z SE_{e_\alpha})\phi(E^{e_\alpha} TE_w) \\ &= \phi\Big(\sum_{\alpha\in A} E^z SE_{e_\alpha} E^{e_\alpha} TE_w\Big) \\ &= E^z(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(ST)E_w, \end{aligned}$$

using the fact that ϕ is homomorphic and norm continuous. □

Remark 4.22. The ideas in the proof above also show that $(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(R(\mathsf{A};\mathsf{H})) \subseteq R(\mathsf{B};\mathsf{H})$ and $(\phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(C(\mathsf{A};\mathsf{H})) \subseteq C(\mathsf{B};\mathsf{H})$.

Lemma 4.23. *Let $\Phi : \mathsf{V} \to \mathsf{W}$ be completely bounded, where the operator space $\mathsf{V} \subseteq B(\mathsf{h}_1)$ and the operator space $\mathsf{W} \subseteq B(\mathsf{h}_2)$. If R, $S \in B(\mathsf{H})$ for some Hilbert space H then*

$$(I_{\mathsf{h}_2} \otimes R)\big((\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})(T)\big)(I_{\mathsf{h}_2} \otimes S) = (\Phi \otimes_{\mathrm{m}} \mathrm{id}_{B(\mathsf{H})})\big((I_{\mathsf{h}_1} \otimes R)T(I_{\mathsf{h}_1} \otimes S)\big)$$

for all $T \in \mathsf{V} \otimes_{\mathrm{m}} B(\mathsf{H})$.

Proof. This follows immediately from the definition. □

Theorem 4.24 (Quantum Itô Product Formula)**.** *Let X and Y be bounded operator processes and let F^* and G be bounded integrable processes such that*

$$X_t^* = X_0^* + \int_0^t F_s^* \,\mathrm{d}\Lambda_s \quad \text{and} \quad Y_t = Y_0 + \int_0^t G_s \,\mathrm{d}\Lambda_s \qquad \text{for all } t \in \mathbb{R}_+.$$

If $H = (F_t\widetilde{Y}_t + \widetilde{X}_t G_t + F_t \Delta_{\mathsf{h}\otimes\mathcal{F}} G_t)_{t\in\mathbb{R}_+}$ is an integrable process then

$$X_t Y_t = X_0 Y_0 + \int_0^t H_s \,\mathrm{d}\Lambda_s \qquad \text{for all } t \in \mathbb{R}_+.$$

Proof. See [16, Corollary 3.16], to which we have added the possibility of having non-zero initial values. □

Remark 4.25. If $s \mapsto H_s$ is only weakly rather than strongly measurable, but with $s \mapsto \langle \xi, H_s\zeta\rangle$ locally integrable for suitable choices of ξ and ζ, then the product process XY will only possess a weak integral representation.

Theorem 4.26. *Let j be a quantum stochastic flow on the unital C^* algebra A, let T be an admissible set and let $\phi : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ be a linear map such that the QSDE (4.1) holds strongly on $\mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T})$ for all $x \in \mathsf{A}_0$.*

Let X and Y be solutions to the multiplier equation (4.5) with generators F and G, respectively, and each with locally bounded norm. Suppose that $F^*\Delta T$, $T\Delta G$, and $F^*\Delta T \Delta G$ are elements of $\mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ for all $T \in \operatorname{im}\phi$, with

$$\widetilde{\jmath}_t(F^*\Delta)\widetilde{\jmath}_t(T) = \widetilde{\jmath}_t(F^*\Delta T), \tag{4.7a}$$

$$\widetilde{\jmath}_t(T)\widetilde{\jmath}_t(\Delta G) = \widetilde{\jmath}_t(T\Delta G) \tag{4.7b}$$

$$\text{and} \qquad \widetilde{\jmath}_t(F^*\Delta)\widetilde{\jmath}_t(T)\widetilde{\jmath}_t(\Delta G) = \widetilde{\jmath}_t(F^*\Delta T \Delta G) \qquad \text{for all } t \in \mathbb{R}_+. \tag{4.7c}$$

Suppose also that $F^*\Delta(x \otimes I_{\widehat{\mathsf{k}}})\Delta G \in \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ for all $x \in \mathsf{A}_0$, with

$$\widetilde{\jmath}_t(F^*\Delta)\widetilde{\jmath}_t\big((x \otimes I_{\widehat{\mathsf{k}}})\Delta G\big) = \widetilde{\jmath}_t(F^*\Delta(x \otimes I_{\widehat{\mathsf{k}}})\Delta G) \qquad \text{for all } t \in \mathbb{R}_+. \tag{4.8}$$

The completely bounded mapping process k on A defined by setting

$$k_t : \mathsf{A} \to B(\mathsf{h} \otimes \mathcal{F});\ x \mapsto X_t^* j_t(x) Y_t \qquad \text{for all } t \in \mathbb{R}_+$$

is such that

$$\mathrm{d}k_t(x) = (\widetilde{k}_t \circ \psi)(x)\,\mathrm{d}\Lambda_t \qquad \text{weakly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \qquad \text{for all } x \in \mathsf{A}_0, \tag{4.9}$$

where

$$\psi : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}});$$
$$x \mapsto (I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + \Delta F)^*\phi(x)(I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + \Delta G) + F^*(x \otimes I_{\widehat{\mathsf{k}}}) + F^*\Delta(x \otimes I_{\widehat{\mathsf{k}}})\Delta G + (x \otimes I_{\widehat{\mathsf{k}}})G. \tag{4.10}$$

If h and k are separable then k satisfies (4.9) strongly.

Proof. Let $x \in \mathsf{A}_0$. We may apply Theorem 4.24 to the processes $t \mapsto j_t(x)$ and Y, noting that the process $t \mapsto j_t(x)^* = j_t(x^*)$ has a stochastic integral representation since $x^* \in \mathsf{A}_0$. Thus

$$j_t(x)Y_t = x \otimes I_{\mathcal{F}} + \int_0^t H_s \,\mathrm{d}\Lambda_s,$$

where

$$H_s = \widetilde{\jmath}_s\big(\phi(x)\big)\widetilde{Y}_s + \big(j_s(x) \otimes I_{\widehat{\mathsf{k}}}\big)\widetilde{\jmath}_s(G)\widetilde{Y}_s + \widetilde{\jmath}_s\big(\phi(x)\big)\Delta\widetilde{\jmath}_s(G)\widetilde{Y}_s$$
$$= \widetilde{\jmath}_s\big(\phi(x) + (x \otimes I_{\widehat{\mathsf{k}}})G + \phi(x)\Delta G\big)\widetilde{Y}_s,$$

where we have used (4.7b), Lemma 4.21 and Lemma 4.23 to combine the three terms. This is valid provided H is an integrable process. To see this, note that $s \mapsto Y_s$ is continuous in the strong operator topology, with locally bounded norm, and so the same is true for $s \mapsto \widetilde{Y}_s$.

Also, the map $s \mapsto \widetilde{\jmath}(A)\xi$ is strongly measurable by Lemma 4.16, again with bounded norm, for any $A \in \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ and $\xi \in \mathsf{h} \otimes \mathcal{F} \otimes \widehat{\mathsf{k}}$.

We next apply Theorem 4.24 once again, to the processes X_t^* and $j_t(x)Y_t$, so that

$$k_t(x) = X_t^* j_t(x) Y_t = x \otimes I_{\mathcal{F}} + \int_0^t L_s \, \mathrm{d}\Lambda_s \qquad \text{weakly,}$$

where

$$L_s := \widetilde{X}_s^* \widetilde{\jmath}_s(F^*)\big(j_s(x)Y_s \otimes I_{\widehat{\mathsf{k}}}\big) + \widetilde{X}_s^* \widetilde{\jmath}_s\big(\theta(x)\big)\widetilde{Y}_s + \widetilde{X}_s^* \widetilde{\jmath}_s(F^*)\Delta\widetilde{\jmath}_s\big(\theta(x)\big)\widetilde{Y}_s$$

and

$$\theta(x) := \phi(x) + (x \otimes I_{\widehat{\mathsf{k}}})G + \phi(x)\Delta G.$$

Lemmas 4.21 and 4.23, together with assumptions (4.7a), (4.8) and (4.7c), now show that

$$L_s = \widetilde{k}_s\big(\psi(x)\big),$$

as required. Although the process $s \mapsto X_s$ is strong operator continuous, this is not guaranteed for $s \mapsto X_s^*$. □

Remark 4.27. Identity (4.8) appears to treat F and G in an asymmetrical fashion. However, Lemma 4.21 implies that

$$\widetilde{\jmath}_t(F^*\Delta)\widetilde{\jmath}_t\big((x \otimes I_{\widehat{\mathsf{k}}})\Delta G\big) = \widetilde{\jmath}_t\big(F^*\Delta(x \otimes I_{\widehat{\mathsf{k}}})\big)\widetilde{\jmath}_t(\Delta G).$$

Using the right-hand side, one can carry out the proof above by first looking at $X_t^* j_t(x)$ and then multiplying on the right by Y_t.

Remark 4.28. If A is a von Neumann algebra and each j_t is ultraweakly continuous then identities (4.7a–c) and (4.8) hold without any need for assumptions on F and G; in this case, the matrix-space lifting $j_t \otimes_{\mathrm{m}} \mathrm{id}_{B(\widehat{\mathsf{k}})}$ is the same as the ultraweak tensor product $j_t \,\overline{\otimes}\, \mathrm{id}_{B(\widehat{\mathsf{k}})}$.

Remark 4.29. For identities (4.7a–c) and (4.8) to hold, it is sufficient that ΔF and ΔG lie in $C(\mathsf{A}; \widehat{\mathsf{k}})$, by Lemma 4.21. If k is finite dimensional then this condition is automatic.

Remark 4.30. (i) Since

$$\mathsf{A} \to B(\mathsf{h} \otimes \mathcal{F});\ x \mapsto F^*(x \otimes I_{\widehat{\mathsf{k}}}) + F^*\Delta(x \otimes I_{\widehat{\mathsf{k}}})\Delta G + (x \otimes I_{\widehat{\mathsf{k}}})G$$

is a bounded map for all F, $G \in \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$, in order to apply Theorem 3.15 to the process k produced by Theorem 4.26, it suffices to find, for all $z, w \in \mathsf{T}$, a norm-dense subspace $\mathsf{A}_{z,w} \subseteq \mathsf{A}_0$ such that the map

$$\mathsf{A}_{z,w} \to \mathsf{A};\ x \mapsto E^{\widehat{z}}(I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + \Delta F)^* \phi(x)(I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + \Delta G)E_{\widehat{w}}$$

is closable and its closure generates a C_0 semigroup.

(ii) Suppose further that $\phi : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ has standard form, as in Definition 4.3, and consider the block-matrix decompositions

$$F = \begin{bmatrix} k_F & m_F \\ l_F & w_F - I_{\mathsf{h}\otimes\mathsf{k}} \end{bmatrix} \qquad \text{and} \qquad G = \begin{bmatrix} k_G & m_G \\ l_G & w_G - I_{\mathsf{h}\otimes\mathsf{k}} \end{bmatrix}. \tag{4.11}$$

A short calculation shows that, for any $x \in \mathsf{A}_0$,

$$\begin{aligned}(I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + \Delta F)^* \phi(x)(I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + \Delta G) \\ = \begin{bmatrix} \tau(x) + l_F^*\delta(x) + \delta^\dagger(x) l_G & \delta^\dagger(x) w_G \\ w_F^*\delta(x) & 0 \end{bmatrix} + \begin{bmatrix} l_F^* \\ w_F^* \end{bmatrix} \big(\pi(x) - x \otimes I_{\mathsf{k}}\big) \begin{bmatrix} l_G & w_G \end{bmatrix}\end{aligned} \tag{4.12}$$

and, with ψ as defined in (4.10), the quantity $\psi(x)$ equals

$$\begin{bmatrix} \tau(x) + l_F^*\delta(x) + l_F^*\pi(x) l_G + \delta^\dagger(x) l_G + k_F^* x + x k_G & \delta^\dagger(x) w_G + l_F^*\pi(x) w_G + x m_G \\ w_F^*\delta(x) + w_F^*\pi(x) l_G + m_F^* x & w_F^*\pi(x) w_G - x \otimes I_{\mathsf{k}} \end{bmatrix}. \tag{4.13}$$

If π extends to a bounded operator from A to $B(\mathsf{h} \otimes \mathsf{k})$ then, as a function of x, the second term in (4.12) defines a bounded operator, so to apply Theorem 3.15 it suffices to find, for all z, $w \in \mathsf{T}$, a norm-dense subspace $\mathsf{A}_{z,w}$ of A_0 such that the map

$$\mathsf{A}_{z,w} \to \mathsf{A};\ x \mapsto \tau(x) + (l_F + w_F E_z)^* \delta(x) + \delta^\dagger(x)(l_G + w_G E_w)$$

is closable with closure that generates a C_0 semigroup.

Remark 4.31. If $F = G$ then the map

$$k_t : \mathsf{A} \to B(\mathsf{h} \otimes \mathcal{F});\ a \mapsto X_t^* j_t(a) X_t$$

produced by Theorem 4.26 is completely positive, for all $t \in \mathbb{R}_+$. This map is unital if and only if X is isometric, and is a $*$-homomorphism if X is co-isometric.

Moreover, if $q(F^*) = 0 = q(F)$, it is not technically difficult to give a direct algebraic proof that the perturbed generator $\psi = \left[\begin{smallmatrix} \tau' & (\delta')^\dagger \\ \delta' & \pi' - \iota \end{smallmatrix}\right]$ defined by (4.13) is a generator in standard form. The equality $q(F^*) = 0$ shows that π' is a $*$-homomorphism, that δ' is a π'-derivation and that τ' satisfies the appropriate cohomological identity; $q(F) = 0$ is only then needed to show w_F is isometric and hence $\psi(1_{\mathsf{A}}) = 0$.

Example 4.32. Continuing the previous Remark, a particular class of generators of interest are those that are "gauge free", that is, the zero map appears in the bottom right corner of the 2×2 block-matrix decomposition. For the generator ϕ of the free flow, this means that $\pi = \iota$, that is, $\pi(x) = x \otimes I_{\mathsf{k}}$ for all $x \in \mathsf{A}_0$. For the perturbed generator ψ to be gauge free as well thus requires that (dropping the subscript on w)

$$w^*(x \otimes I_{\mathsf{k}}) w = x \otimes I_{\mathsf{k}} \qquad \text{for each } x \in \mathsf{A}_0. \tag{4.14}$$

Setting $x = 1_{\mathsf{A}}$ shows that w must be isometric, which is one of the conditions that must be satisfied to have $q(F) = 0$, a necessary condition for X to be isometric. Note that (4.14) holds whenever w is an isometric element of $(\mathsf{A} \otimes I_{\mathsf{k}})' \cap \big(\mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k})\big) = Z(\mathsf{A}) \otimes_{\mathrm{m}} B(\mathsf{k})$, where $Z(\mathsf{A})$ is the centre of A. If w is co-isometric as well, then belonging to $Z(\mathsf{A}) \otimes_{\mathrm{m}} B(\mathsf{k})$ is a necessary and sufficient condition on w to ensure that ψ is gauge free.

5 Examples

5.1 Weyl perturbations

Let j be a quantum stochastic flow on the C^* algebra A such that

$$j_0(x) = x \otimes I_{\mathcal{F}} \quad \text{and} \quad \mathrm{d}j_t(x) = \widetilde{\jmath}_t\big(\phi(x)\big)\,\mathrm{d}\Lambda_t \qquad \text{strongly on } \mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{T}) \tag{5.1}$$

for all $x \in \mathsf{A}_0$, where A_0 is a norm-dense $*$-subalgebra of A.

Let $F = I_{\mathsf{h}} \otimes C$ for

$$C = \begin{bmatrix} \mathrm{i}h - \frac{1}{2}\|c\|^2 & -\langle U^*c| \\ |c\rangle & U - I_{\mathsf{k}} \end{bmatrix} \in B(\widehat{\mathsf{k}}),$$

where $h \in \mathbb{R}$, $c \in \mathsf{k}$ and $U \in B(\mathsf{k})$ is unitary. By Remark 4.15(iv) and Proposition 4.14 with multiplier generator $F = I_{\mathsf{h}} \otimes C$, there exists a unitary operator process X such that

$$X_t = I_{\mathsf{h} \otimes \mathcal{F}} + \int_0^t X_s \otimes C\,\mathrm{d}\Lambda_s \qquad \text{for all } t \in \mathbb{R}_+.$$

Since

$$\Delta F E_{\widehat{z}} = I_{\mathsf{h}} \otimes |\xi\rangle \in \mathsf{A} \underline{\otimes} |\widehat{\mathsf{k}}\rangle \qquad \text{for all } z \in \mathsf{k},$$

where $\xi = (0, c{+}Uz{-}z) \in \widehat{\mathsf{k}}$, it follows that $\Delta F \in C\big(\mathsf{A}; \widehat{\mathsf{k}}\big)$. Thus Remark 4.29 and Theorem 4.26 give a strongly continuous $*$-homomorphic mapping process k.

If the free-flow generator $\phi : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B\big(\widehat{\mathsf{k}}\big)$ has standard form, so that $\phi = \left[\begin{smallmatrix} \tau & \delta^\dagger \\ \delta & \pi - \iota \end{smallmatrix}\right]$, then the perturbed generator ψ of k is such that $\psi(x)$ equals

$$\begin{bmatrix} \tau(x) + E^c\delta(x) + E^c\pi(x)E_c + \delta^\dagger(x)E_c - \|c\|^2 x & \delta^\dagger(x)(I_{\mathsf{h}} \otimes U) + E^c\pi(x)(I_{\mathsf{h}} \otimes U) - xE^{U^*c} \\ (I_{\mathsf{h}} \otimes U)^*\delta(x) + (I_{\mathsf{h}} \otimes U)^*\pi(x)E_c - E_{U^*c}x & (I_{\mathsf{h}} \otimes U)^*\pi(x)(I_{\mathsf{h}} \otimes U) - x \otimes I_{\mathsf{k}} \end{bmatrix}$$

for all $x \in \mathsf{A}_0$. In particular, if $\pi = \iota$, so that $\phi = \left[\begin{smallmatrix} \tau & \delta^\dagger \\ \delta & 0 \end{smallmatrix}\right]$, then $\psi = \left[\begin{smallmatrix} \tau' & (\delta')^\dagger \\ \delta' & 0 \end{smallmatrix}\right]$, where

$$\tau' : \mathsf{A}_0 \to \mathsf{A};\ x \mapsto \tau(x) + E^c\delta(x) + \delta^\dagger(x)E_c \quad \text{and} \quad \delta' : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} |\mathsf{k}\rangle;\ x \mapsto (I_{\mathsf{h}} \otimes U)^*\delta(x).$$

In particular, we see that

$$\psi_{\widehat{w}}^{\widehat{z}} = \phi_{\widehat{c+Uw}}^{\widehat{c+Uz}} \qquad \text{for all } z, w \in \mathsf{k}.$$

If j has a strongly continuous vacuum-expectation semigroup then Proposition 3.16 applies and shows that each $\phi_{\widehat{w}}^{\widehat{z}}$ is closable and has an extension that generates a C_0 semigroup. If in fact it is the closure of $\phi_{\widehat{w}}^{\widehat{z}}$ that is the semigroup generator, and if $c + Uz \in \mathsf{T}$ for all $z \in \mathsf{T}$, then k is a Markovian cocycle by Theorem 3.15. These conditions are satisfied by the cocycles constructed in Theorems 3.9 and 3.12 of [7] where $\mathsf{T} = \mathsf{k}$, as shown by a combination of Theorem 3.16 and Lemma 2.14 of that paper. These results show that A_0 is a core for the generators of all of the associated semigroups.

5.2 The quantum exclusion process

Let $\mathsf{h} = \mathcal{F}_-\big(\ell^2(I)\big)$, the Fermionic Fock space over $\ell^2(I)$, where I is a non-empty set, and let b_i and b_i^* be the annihilation and creation operators at site $i \in I$, respectively, so that

$$\{b_i, b_j\} = 0 \qquad \text{and} \qquad \{b_i, b_j^*\} = \mathbb{1}_{i=j} \qquad \text{for all } i, j \in I.$$

The *CAR algebra* A is the norm closure of A_0, the $*$-algebra generated by $\{b_i : i \in I\}$, in $B(\mathsf{h})$.

The quantum exclusion process was introduced by Rebolledo [25], and the associated process constructed in [7] under certain assumptions discussed below. The two inputs, given the choice of I, are functions $\eta : I \to \mathbb{R}$ and $\alpha : I \times I \to \mathbb{C}$. The former gives the energy η_i at each site $i \in I$, and $\alpha_{i,j}$ is an amplitude from site i to site j. We set

$$t_{i,j}^{\alpha} := \alpha_{i,j} b_j^* b_i \qquad \text{and} \qquad \delta_{i,j}^{\alpha} : \mathsf{A} \to \mathsf{A};\ x \mapsto [t_{i,j}^{\alpha}, x] \qquad (i, j \in I).$$

Let k have orthonormal basis $\{f_{i,j} : i, j \in I\}$, so that $\mathsf{k} \cong l^2(I \times I)$. We can assemble the set of operators $\{t_{i,j}^{\alpha} : i, j \in I\}$ into a column operator and the set of derivations $\{\delta_{i,j}^{\alpha} : i, j \in I\}$ into an associated ι-derivation:

$$t_\alpha := \sum_{i,j \in I} t_{i,j}^{\alpha} \otimes |f_{i,j}\rangle \tag{5.2}$$

$$\text{and} \quad \delta_\alpha : \mathsf{A}_0 \to \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k});\ x \mapsto \sum_{i,j \in I} \delta_{i,j}^{\alpha}(x) \otimes |f_{i,j}\rangle = t_\alpha x - (x \otimes I_{\mathsf{k}}) t_\alpha. \tag{5.3}$$

If the series in (5.2) converges to give a bounded operator from h to $\mathsf{h} \otimes \mathsf{k}$ then $t_\alpha \in \mathsf{A} \otimes_{\mathrm{m}} |\mathsf{k}\rangle$; moreover, δ_α is a well defined ι-derivation, and the domain of δ_α can be extended to all of A. However, the standing assumption in [7] does not necessarily give such convergence. Instead, the following *finite-valence* and *symmetry* assumptions were made:

$$\{j \in I : \alpha_{i,j} \neq 0\} \text{ is finite for all } i \in I \qquad \text{and} \qquad |\alpha_{i,j}| = |\alpha_{j,i}| \text{ for all } i, j \in I. \tag{5.4}$$

Under these assumptions it turns out that $\delta_\alpha(x)$ is well defined for each $x \in \mathsf{A}_0$ since only finitely many of the terms in the series (5.3) are non-zero, even if t_α is not a bounded operator.

The generator of the semingroup is then given by

$$\begin{aligned}\tau_{\alpha,\eta} : \mathsf{A}_0 \to \mathsf{A}_0;\ x \mapsto \mathrm{i} \sum_{i \in I} \eta_i [b_i^* b_i, x] - \frac{1}{2} \sum_{i,j \in I \times I} \big((t_{i,j}^{\alpha})^* [t_{i,j}^{\alpha}, x] + [x, (t_{i,j}^{\alpha})^*] t_{i,j}^{\alpha} \big) \\ = \mathrm{i}[H, x] - \tfrac{1}{2} t_\alpha^* \delta_\alpha(x) - \tfrac{1}{2} \delta_\alpha^\dagger(x) t_\alpha,\end{aligned}$$

where $H := \sum_{i\in I} \eta_i b_i^* b_i$. Again, the series for H may not converge, but the commutator $[H, x]$ is well defined, as are the products $t_\alpha^* \delta_\alpha(x)$ and $\delta_\alpha^\dagger(x) t_\alpha$, courtesy of (5.4).

We can now write

$$\phi = \begin{bmatrix} \tau_{\alpha,\eta} & \delta_\alpha^\dagger \\ \delta_\alpha & 0 \end{bmatrix} : \mathsf{A}_0 \to \mathsf{A}_0 \underline{\otimes} B(\widehat{\mathsf{k}}).$$

This map has standard form, and under a variety of hypotheses [7, Examples 5.11–13] it was shown that there exists a quantum stochastic flow j with ϕ as its generator.

For the remainder of this section we will assume that such a flow j exists, and show how the methods of this paper can allow us to go beyond the assumptions (5.4).

Let $\beta : I \times I \to \mathbb{C}$, and suppose that t_β defined through (5.2) is a well defined element of $\mathsf{A} \otimes_{\mathrm{m}} |\mathsf{k}\rangle$ such that $t_\beta^* t_\beta \in \mathsf{A}$. Choose $h \in \mathsf{A}$ such that $h = h^*$ and set

$$F := \begin{bmatrix} \mathrm{i}h - \frac{1}{2} t_\beta^* t_\beta & t_\beta^* \\ -t_\beta & 0 \end{bmatrix}.$$

The strongly continuous operator process $X = X^F$ given by Proposition 4.14 is isometric. Furthermore, since $\Delta F = \left[\begin{smallmatrix} 0 & 0 \\ -t_\beta & 0 \end{smallmatrix}\right] \in C(\mathsf{A}; \widehat{\mathsf{k}})$, Remark 4.29 makes it clear that Theorem 4.26 may be applied, when I is countable, to obtain a completely positive unital mapping process k with generator

$$\psi = \begin{bmatrix} \tau' & (\delta')^\dagger \\ \delta' & 0 \end{bmatrix}, \tag{5.5}$$

where

$$\tau'(x) = \tau_{\alpha,\eta}(x) - t_\beta^* \delta_\alpha(x) + t_\beta^*(x \otimes I_{\mathsf{k}}) t_\beta - \delta_\alpha^\dagger(x) t_\beta - \tfrac{1}{2}\{t_\beta^* t_\beta, x\} - \mathrm{i}[h, x]$$

$$\text{and} \quad \delta'(x) = \delta_\alpha(x) - (x \otimes I_{\mathsf{k}}) t_\beta + t_\beta x.$$

for all $x \in \mathsf{A}_0$. In particular for any choice of β we have $\delta' = \delta_{\alpha+\beta}$. Furthermore, if the series

$$t_\alpha^* \delta_\beta(x) = \sum_{i,j\in I} \overline{\alpha_{i,j}} \beta_{i,j} b_i^* b_j [x, b_j^* b_i] \tag{5.6}$$

is convergent in the weak operator topology for all $x \in \mathsf{A}_0$ then the expression defining τ' above can also be rigorously manipulated to show that $\tau' = \tau_{\alpha+\beta,\eta}$. Thus k is a process with a generator of the same structure as j, but the amplitudes have been changed. This can have two obvious benefits:

(i) the symmetry condition $|\alpha_{i,j} + \beta_{i,j}| = |\alpha_{j,i} + \beta_{j,i}|$ need not apply;

(ii) the finite-valence condition need not apply to $\alpha + \beta$.

The cost, currently, for circumventing these restrictions imposed in our earlier work [7] is that it is not yet known if the resulting process k is multiplicative.

As an example of conditions that ensure that the various series above behave as required, we give one set of sufficient hypotheses.

Theorem 5.1. *Let $\eta \in l^\infty(I)$, $\alpha \in l^\infty(I \times I)$ and $\beta \in l^1(I \times I)$, where I is a countable set and α satisfies (5.4). The process k with generator (5.5) exists and satisfies the QSDE (4.1) strongly on $\mathsf{h} \underline{\otimes} \mathcal{E}(\mathsf{k})$ for all $x \in \mathsf{A}_0$.*

Proof. Existence of j is dealt with in Example 5.11 of [7]. Since α is bounded and β is summable, the series (5.6) is norm convergent and so the results of Sections 4.2 and 4.3 apply. □

Remark 5.2. Only very minimal assumptions have been made regarding the graph with I as the set of vertices and an edge between i and j whenever $\alpha_{i,j} + \beta_{i,j} \neq 0$. A more detailed study of this graph would undoubtedly allow less restrictive assumptions to be imposed on the perturbation function β.

Remark 5.3. An alternative approach to constructing quantum exclusion processes has been developed in [23], based on an analysis of the associated semigroups. No assumption of symmetry is made on the amplitudes, but it is assumed that $I = \mathbb{Z}^d$ and that $\alpha_{i,j} \neq 0$ only for sites i and j within a fixed distance of each other. As in this paper, it is not yet known if the resulting cocycle in [23] is multiplicative.

5.3 Flows on universal C^* algebras

Definition 5.4. Let A be a unital C^* algebra A and let $\{a_i : i \in I\}$ be a subset of A. We let W denote the set of all words in the elements $\{a_i, a_i^* : i \in I\}$, so that $\mathsf{A}_0 = \operatorname{lin} \mathsf{W}$ is the $*$-subalgebra generated by this subset. The set W is said to *generate* A if A_0 is norm dense in A.

These generators *satisfy the relations* $\{p_k : k \in K\}$ if each p_k is a complex polynomial in the non-commuting indeterminates $\langle X_i, X_i^* : i \in I\rangle$ and the algebra element $p_k(a_i, a_i^* : i \in I)$ obtained from p_k by replacing each X_i by a_i and X_i^* by a_i^* is equal to 0 for all $k \in K$.

A generator a_i is called *balanced* if in each relation p_k the difference between the number of instances of a_i and the number of instances of a_i^* in each monomial making up p_k is constant.

Let the unital C^* algebra A have generators $\{a_i : i \in I\}$ that satisfy the relations $\{p_k : k \in K\}$. Then A is *universal* if, given any unital C^* algebra B containing a set of elements $\{b_i : i \in I\}$ which satisfies the relations $\{p_k : k \in K\}$, that is, $p_k(b_i, b_i^* : i \in I) = 0$ for all $k \in K$, there exists a unique $*$-homomorphism $\pi : \mathsf{A} \to \mathsf{B}$ such that $\pi(a_i) = b_i$ for all $i \in I$.

Example 5.5. In [7] we considered flows on the universal rotation algebra, which is the universal algebra generated by the unitaries U, V and Z such that

$$UV = ZVU, \quad UZ = ZU \quad \text{and} \quad VZ = ZV.$$

If we label the generators as $a_1 = U$, $a_2 = V$ and $a_3 = Z$ then the full list of relations are

$$p_1 = X_1X_1^* - 1, \quad p_2 = X_1^*X_1 - 1, \quad p_3 = X_2X_2^* - 1, \quad p_4 = X_2^*X_2 - 1, \quad p_5 = X_3X_3^* - 1,$$

$$p_6 = X_3^*X_3 - 1, \quad p_7 = X_1X_2 - X_3X_2X_1, \quad p_8 = X_1X_3 - X_3X_1, \quad p_9 = X_2X_3 - X_3X_2.$$

The relations $p_1, \ldots, p_6$ encode the unitarity of the generators. Note that U and V are both balanced, but Z is not courtesy of relation p_7.

Lemma 5.6. *Let* A *be a universal* C^* *algebra generated by* $\{a_i : i \in I\}$ *and let* W *and* A_0 *be as in* Definition 5.4. *Suppose that the generating element* a_j *is balanced. There is a* C_0 *group of automorphisms* α^j *with generator* $\mathrm{i}d_j$ *such that*

(i) *the domain* $\operatorname{dom} d_j$ *is a* $*$*-subalgebra containing* A_0,

(ii) *the derivation* d_j *is skew symmetric, that is,* $d_j^\dagger = -d_j$, *and*

(iii) *we have* $d_j(b) = n_j(b)b$ *for each* $b \in \mathsf{W}$, *where*

$$n_j(b) := \textit{number of copies of } a_j \textit{ in } b - \textit{number of copies of } a_j^* \textit{ in } b.$$

Proof. For each $t \in \mathbb{R}$, let

$$b_i := \begin{cases} a_i & \text{if } i \neq j, \\ e^{\mathrm{i}t} a_j & \text{if } i = j. \end{cases} \tag{5.7}$$

Since a_j is balanced it follows that $\{b_i : i \in I\}$ is a set of elements in A that satisfy the same relations as the original set of generators, and moreover this new set also generates A. Universality implies that is an automorphism α_t^j of A such that $\alpha_t^j(a_i) = b_i$ for all i. It is easy to see that $\alpha_t^j(b) = e^{\mathrm{i}tn_j(b)}b$ for all $b \in \mathsf{W}$ and $t \in \mathbb{R}$. Since $\|\alpha_t^j\| = 1$ for all t, it follows by linearity and continuity that $(\alpha_t^j)_{t \in \mathbb{R}}$ is a C_0 group of automorphisms.

That $\operatorname{dom} d_j$ is a $*$-subalgebra of A and d_j is skew symmetric follows from the fact that α^j is a group of automorphisms. Moreover, it is immediate that $\mathsf{W} \subseteq \operatorname{dom} d_j$, with $d_j(b) = n_j(b)b$ for each $b \in \mathsf{W}$, and so $\mathsf{A}_0 \subseteq \operatorname{dom} d_j$. □

Theorem 5.7. *Let* A *be a universal* C^* *algebra generated by the set* $\{a_i : i \in I\}$, *let* $\{a_j : j \in J\}$ *be a subset of balanced generators, where* $J \subseteq I$, *and let* $\mathsf{k} := \ell^2(J)$ *with the standard orthonormal basis* $\{e_j : j \in J\}$. *For any choices of constants* $\{c_j \in \mathbb{C} : j \in J\}$, *the following defines a generator* $\phi : \mathsf{A}_0 \to \mathsf{A}_0 \otimes B(\mathsf{k})$ *in standard form according to (4.2)*:

$$\delta(x) = \sum_{j \in J} c_j d_j(x) \otimes |e_j\rangle, \qquad \tau = -\tfrac{1}{2} \sum_{j \in J} |c_j|^2 d_j^2 \quad \textit{and} \quad \pi = \iota, \tag{5.8}$$

where d_j *is as defined in* Lemma 5.6. *There is a weakly multiplicative strong solution* j *to (4.1) for this generator* ϕ. *If, in addition, the generators are all isometries then* j *is* $*$*-homomorphic.*

Proof. The series for $\delta(x)$ and $\tau(x)$ both have only finitely many non-zero terms, so are well defined, since $d_j(a_k) = \mathbb{1}_{j=k} \mathrm{i} a_k$. Hence ϕ is a generator is standard form, by Example 4.6.

Furthermore, if $j \in J$ then $\phi(a_j) = a_j \otimes T_j$ for

$$T_j := -\tfrac{1}{2}|c_j|^2 |f_0\rangle\langle f_0| + c_j |f_j\rangle\langle f_0| - \overline{c_j} |f_0\rangle\langle f_j| \in B(\widehat{\mathsf{k}}),$$

where $f_0 = (1,0)$ and $f_j = (0, e_j) \in \widehat{\mathsf{k}}$. On the other hand, if $i \in I \setminus J$ then $\phi(a_i) = 0$. It follows from [7, Corollary 2.12, Theorem 3.5 and Lemma 3.7] that we can solve (4.1) to find a weakly multiplicative solution j. If the generators are all isometries then [7, Theorem 3.12] allows us to complete the proof. □

Example 5.8. The non-commutative torus is determined by a choice of $\lambda \in \mathbb{T}$. Given such λ, we define unitary operators U and V on $\ell^2(\mathbb{Z}^2)$ by setting

$$(Uu)_{m,n} := u_{m+1,n} \qquad \text{and} \qquad (Vu)_{m,n} := \lambda^m u_{m,n+1} \qquad \text{for all } u \in \ell^2(\mathbb{Z}^2) \text{ and } m, n \in \mathbb{Z},$$

and let A_0 be the $*$-algebra generated by U and V. Note that $UV = \lambda VU$ and A, the norm closure of A_0 in $B\big(\ell^2(\mathbb{Z}^2)\big)$, is then a concrete realisation of the non-commutative torus with parameter λ, which is a universal C^* algebra. Lemma 5.6 gives existence of the derivations

$$d_1 : \mathsf{A}_0 \to \mathsf{A}_0;\ U^m V^n \mapsto m U^m V^n \qquad \text{and} \qquad d_2 : \mathsf{A}_0 \to \mathsf{A}_0;\ U^m V^n \mapsto n U^m V^n,$$

and the free flow generated by ϕ defined in Theorem 5.7 is the one discussed in [7, Theorem 6.9].

Example 5.9. For each $N \in \mathbb{N} \cup \{\infty\}$, the Cuntz algebra $\mathcal{O}_N$ is the C^* algebra generated by isometries $\{s_i\}_{i=1}^N$ that satisfy the additional relation $\sum_{i=1}^N s_i s_i^* = 1$. For finite N, a concrete realisation can be given as follows: let $\mathsf{h} = \ell^2(\mathbb{Z})$ with standard orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ and define s_i by continuous linear extension of the map $s_i : e_n \mapsto e_{nN+i}$. A similar construction is possible in the case when $N = \infty$. As has been known since its introduction, all realisations of $\mathcal{O}_N$ are mutually isomorphic [10, Theorem 1.12] and so it is universal. Thus Theorem 5.7 applies to give a flow on $\mathcal{O}_N$, where the dimension of the multiplicity space is N.

Example 5.10. More recent examples of universal C^* algebras are the multitude of non-commutative spheres studied in [3]. The following are all non-commutative examples for which all of the generators $\{z_i\}_{i=1}^N$ are balanced:

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\Big(\{z_i\}_{i=1}^N : \sum_{i=1}^N z_i z_i^* = \sum_{i=1}^N z_i^* z_i = 1\Big);$$

$$C(\overline{S}^{N-1}_{\mathbb{C}}) = C(S^{N-1}_{\mathbb{C},+})/\langle \alpha\beta = -\beta\alpha \text{ for all distinct } a, b \in \{z_i\},\ \alpha\beta = \beta\alpha \text{ otherwise}\rangle;$$

$$C(S^{N-1}_{\mathbb{C},**}) = C(S^{N-1}_{\mathbb{C},+})/\big\langle abc = cba \ \text{ for all } a, b, c \in \{z_i, z_i^*\}\big\rangle;$$

$$C(\overline{S}^{N-1}_{\mathbb{C},**}) = C(S^{N-1}_{\mathbb{C},+})/\langle \alpha\beta\gamma = -\gamma\beta\alpha \text{ for all distinct } a, b, c \in \{z_i\},\ \alpha\beta\gamma = \gamma\beta\alpha \text{ otherwise}\rangle.$$

The first of these is the *complex free sphere*. If we also include the relations $[z_i, z_j] = 0$ for all i and j then we have a commutative C^* algebra that is isomorphic to the algebra of continuous functions on the complex sphere, but in the free sphere all commutativity has been eradicated.

The second algebra is the *twisted* version of the sphere, obtained by taking a quotient of the free sphere. The notation with a and α means that if $a = z_i$ then α stands for either of z_i or z_i^*. Thus we have the imposed the following additional relations to those satisfied by the free sphere: $z_i z_i^* = z_i^* z_i$ and, for $i \neq j$, $z_i z_j = -z_j z_i$ and $z_i z_j^* = -z_j^* z_i$. The third and fourth algebras are the *half-liberations* of $S^{N-1}_{\mathbb{C}}$ and $\overline{S}^{N-1}_{\mathbb{C}}$, respectively.

In [3] there are also many real spheres studied, which have generators $x_1, \ldots, x_N$ that are self adjoint and satisfy $\sum_{i=1}^N x_i^2 = 1$. The relation $x_i = x_i^*$ ensures that none of the generators are balanced, and so Lemma 5.6 is not applicable.

Proposition 5.11. *For each of the four spheres from* Example 5.10 *the gauge-free generator* ϕ *defined by (5.8) is well defined and the solution* j *to (4.1) for this generator is* $*$*-homomorphic in the strong sense.*

Proof. Following the proof of Theorem 5.7, we get a weakly multiplicative, strong solution by applying the results from [7]. However, since the generators z_i are not isometries, we cannot apply [7, Theorem 3.12] directly. This problem is overcome by noting that since j is unital and weakly multiplicative we have that

$$0 \leqslant \sum_{i=1}^{N} \langle j_t(z_i)\xi, j_t(z_i)\xi\rangle = \langle \xi, j_t\Big(\sum_{i=1}^{N} z_i^* z_i\Big)\xi\rangle = \|\xi\|^2 \qquad \text{for any } t \in \mathbb{R}_+ \text{ and } \xi \in \mathsf{h} \underline{\otimes} \mathcal{E}.$$

Consequently each map $j_t(z_i)$ is contractive, and we can consider for fixed t the C^* subalgebra of $B(\mathsf{h} \otimes \mathcal{F})$ generated by these bounded operators. By universality, this algebra is isomorphic to the relevant sphere, and so j_t is indeed a well-defined $*$-homomorphism. □

Remark 5.12. Let A be a C^* algebra as in Lemma 5.6, with generators $a_1, \ldots, a_N$ being balanced, and in addition possibly having other generators. Let $\alpha^1, \ldots, \alpha^N$ be the C_0 groups associated to the given balanced generators. Together these define an action of $\mathbb{R}^N$ on A:

$$\alpha_{\mathbf{t}} := \alpha^1_{t_1} \circ \cdots \circ \alpha^N_{t_N} \quad \text{for } \mathbf{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N,$$

since $\alpha^i_s \circ \alpha^j_t = \alpha^j_t \circ \alpha^i_s$ for all i, j, s and t. Indeed, periodicity of the groups actually implies that we have an action of the N-torus $\mathbb{T}^N$ on A.

Let $\mathbf{w} = (w^1, \ldots, w^N)$ be an N-dimensional Wiener process, and pick $c_1, \ldots, c_N \in \mathbb{C}$. We can randomise the group action to define a mapping process l as follows:

$$l_t(x) := \alpha_{\mathbf{w}_t}(x) = \big(\alpha^1_{|c_1| w^1_t} \circ \cdots \circ \alpha^N_{|c_N| w^N_t}\big)(x) \qquad \text{for any } x \in \mathsf{A}. \tag{5.9}$$

Furthermore, if j is the flow on A from Theorem 5.7 then, writing $c_j = \mathrm{i}|c_j|e^{\mathrm{i}\theta_j}$ for $\theta_j \in [0, 2\pi)$, we have that

$$\mathrm{d}j_t(x) = j_t\Big(-\tfrac{1}{2}\sum_{i=1}^{N} |c_j|^2 d_j^2(x)\Big)\,\mathrm{d}t + \sum_{j=1}^{N} j_t\big(\mathrm{i}|c_j| d_j(x)\big)\big(e^{\mathrm{i}\theta_j}\,\mathrm{d}(A^j)^*_t + e^{-\mathrm{i}\theta_j}\,\mathrm{d}(A^j)_t\big)$$

where $x \in \mathsf{A}_0$ and A^j is the jth component of the annihilation operator with respect to the given basis $\{e_1, \ldots, e_N\}$ of k. There is a natural identification of the Fock space $\mathcal{F}$ with the L^2-space for $\mathbf{w}$ (see [16]), and moreover each operator process $B^j_t := e^{\mathrm{i}\theta_j}(A^j)^*_t + e^{\mathrm{i}\theta_j}(A^j)_t$ can then be viewed as a realisation of the corresponding component of $\mathbf{w}$. Applying the usual Itô Lemma to (5.9), and using these identifications, it follows that $l_t(x) \cong j_t(x)$, so that the flows arising from Theorem 5.7 have a classical origin.

Example 5.13. Let ϕ be the gauge-free generator in standard form from Theorem 5.7 and let j be the free flow generated by ϕ. Suppose $F = \begin{bmatrix} k & -l^* \\ l & 0 \end{bmatrix} \in \mathsf{A} \otimes_{\mathrm{m}} B(\widehat{\mathsf{k}})$ is a gauge-free multiplier generator, where $k \in \mathsf{A}$ and $l \in \mathsf{A} \otimes_{\mathrm{m}} B(\mathsf{k})$. To satisfy the conditions $q(F) = q(F^*) = 0$, we require that $k + k^* + l^* l = 0$. If X is the solution to (4.5) and $k_t = X_t^* j_t(\cdot) X_t$ then k is a weak solution to the QSDE (4.9) for the generator ψ, where

$$\psi(x) = \begin{bmatrix} \tau(x) + l^*\delta(x) + l^*(x \otimes I_{\mathsf{k}})l + \delta^\dagger(x)l + k^*x + xk & \delta^\dagger(x) + l^*(x \otimes I_{\mathsf{k}}) - xl^* \\ \delta(x) + (x \otimes I_{\mathsf{k}})l - lx & 0 \end{bmatrix}.$$

If the set $\{a_i : i \in I\}$ of generators of A is countable then k is separable, and we may assume that h is also separable [24, Corollary 3.7.5]. The process k is then a strong solution to (4.9).

Letting $l_j := \big(I_{\mathsf{h}} \otimes \langle e_j|\big) l \in \mathsf{A}$, we have that

$$\delta(x) + (x \otimes I_{\mathsf{k}})l - lx = \sum_{j \in J} \big(c_j d_j(x) + d_{l_j}(x)\big),$$

where

$$d_r(x) := [x, r] = xr - rx \qquad \text{for any } r \in \mathsf{A}.$$

Feynman–Kac perturbation techniques similar to those developed here were employed in [9] as a means of constructing possible Laplacians for the non-commutative torus, in which the components $c_j d_j + d_{l_j}$ should be thought of as Dirac operators. More precisely, in [9] they studied operators on the Hilbert space $L^2(\mathsf{A})$ arising from the unique trace on the non-commutative torus. To fit into the framework of non-commutative geometry, the operators on this space arising from these derivations ought to be self adjoint, and so the component derivations should be symmetric. This forces the choice of $c_j = \mathrm{i}\beta_j$ for some $\beta_j \in \mathbb{R}$ when looking at the generator of the free flow. For the perturbed generator, since $(d_r)^\dagger = -d_{r^*}$, it is appropriate to make the choice $l_j = \mathrm{i}n_j$, where $n_j \in \mathsf{A}$ is self adjoint.

Under the assumption that $c_j = \mathrm{i}\beta_j$, where $\beta_j \in \mathbb{R}$, the unperturbed semigroup generator

$$\tau = -\frac{1}{2} \sum_{j \in J} (\beta_j d_j)^2.$$

If

$$\tau'(x) = \tau(x) + l^*\delta(x) + l^*(x \otimes I_{\mathsf{k}})l + \delta^\dagger(x)l + k^*x + xk \qquad \text{and} \qquad d_j' = \mathrm{i}(\beta_j d_j + d_{n_j})$$

then we can ask if the perturbed semigroup generator satisfies the analogous equation:

$$\tau' = -\tfrac{1}{2} \sum_{j \in J} (d_j')^2 = -\tfrac{1}{2} \sum_{j \in J} (\beta_j d_j + d_{n_j})^2. \tag{5.10}$$

If J is finite then equation (5.10) holds if each $n_j \in \mathsf{A}_0$ is self adjoint and we choose

$$k = -\tfrac{1}{2} \sum_{j \in J} \big(\beta_j d_j(n_j) + n_j^2\big). \tag{5.11}$$

The requirement that $q(F) = 0$ ensures that the value of $k + k^*$ must be $-l^* l$, and this is consistent with (5.11). However, we can also see from (5.11) that $k - k^* = \mathrm{i} \sum_j \beta_j d_j(n_j)$, so the imaginary part of k is not completely arbitrary.

6 References

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